MOTIVIC MULTIPLICATIVE STRUCTURES

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Abstract. This is a reading proposal with topics on motivic multiplicative structures and infinite loop spaces.

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1. Overview of motivic multiplicative structure

The following diagram presents the comparison between classical multiplicative structures in terms of operads and motivic multiplicative structures in terms of multiplicative transfers.

$$
\begin{tikzcd}
\substack{\mathrm{Span}(\mathcal{F}in, \mathrm{inj}, \mathrm{all}) \\ \text{universal commutative monoid}} \arrow[d, "\text{parametrize}"] & \boxed{\text{intrinsic symmetry}} \arrow[l, "\text{presented as}"'] \arrow[r, "\text{presented as}"] \arrow[d, "\text{encoded in}"] & \substack{\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \\ \text{universal normed category}} \arrow[d, "\text{parametrize}"'] \\
\substack{\mathbb{E}_\infty\text{-structure} \\ \mathbb{E}_\infty\text{-operad action}} \arrow[d, "\text{pass to homotopy}"] & \text{multiplicative coherence data} \arrow[l, "\text{encoded in}"'] \arrow[r, "\text{encoded in}"] & \text{motivic normed structure} \arrow[d, dashed, "\text{take motivic colimits}"'] \\
\substack{\mathbb{H}_\infty\text{-structure} \\ \mathbb{E}_\infty\text{-operad homotopy action}} \arrow[d, "\text{pass to cohomology}"] & & \text{motivic extended powers} \arrow[d, dashed, "\text{pass to cohomology}"'] \\
\text{power operations} & & \text{motivic power operations}
\end{tikzcd}
$$

Figure 1. Multiplicative structures in different contexts

Remark 1.1. Note that

$$\mathcal{F}\mathrm{un}(\mathrm{Span}(\mathcal{F}in, \mathrm{inj}, \mathrm{all}), \mathcal{C}) \simeq \mathcal{F}\mathrm{un}(\mathcal{F}in_*, \mathcal{C}),$$

where $\mathcal{C}$ is an ∞-category with finite products [BH21, Proposition C.1.]. Thus we may conclude that $\mathrm{Span}(\mathcal{F}in, \mathrm{inj}, \mathrm{all})$ is the universal commutative monoid. The process from $\mathcal{F}in_*$ to $\mathbb{E}_\infty$-structure is subtle. More precisely, we should take the nerve $N(\mathcal{F}in_*)$ of $\mathcal{F}in_*$. I think should we regard $N(\mathcal{F}in_*)$ as a kind of universal category of operators, and we may produce an $\mathbb{E}_\infty$-operad from this perspective, but I do not know how to do it rigorously. The key point is that there should a translation between the ordinary-categorically operadic formulation and the ∞-categorical formulation for multiplicative structure. Moreover, may we need to adapt the notion of categories of operators to algebraic geometry from this perspective.

2. Norms in motivic stable homotopy theory

In this chapter, we assume that all the based schemes are quasi-compact and quasi-separated (we need this assumption, otherwise some categories over S are not compactly generated). Let Sm_S be the category of smooth schemes of finite type over S.

Theorem 2.1. Let $p : T \to S$ be an integral and universally open morphism. Then there is a symmetric monoidal functor called **multiplicative transfer**

$$p_\otimes \colon \mathcal{H}_\bullet(T) \to \mathcal{H}_\bullet(S)$$

such that

(1) Sifted colimits is preserved by $p_\otimes$.
(2) p_* is extended by $p_\otimes$ i.e. $p_\otimes(Y_+) \simeq (p_*Y)_+$ for $Y \in \mathrm{Sm}_S$.
(3) If $p \colon S \times \{\underline{n}\} \to S$ is the trivial projection (we may also write it into a fold map $S^{\sqcup n} \to S$), then $p_\otimes$ is the n-fold smash product.

If p is a finite étale , then we can extend $p_\otimes$ to stable motivic homotopy categories

$$p_\otimes \colon \mathcal{SH}(T) \to \mathcal{SH}(S).$$

This section is devoted to sketch the construction of this multiplicative transfers.
First, we recall the construction of stable motivic homotopy category $\mathcal{SH}(S)$ over S.

$$\begin{tikzcd}
\mathrm{Sm}_S \arrow[r, "\text{add based points}", "X \mapsto X \sqcup S"'] \arrow[d, "\text{sifted cocompletion}"'] & \mathrm{Sm}_{S+} \arrow[d, "\text{sifted cocompletion}"] \\
\mathcal{P}_\Sigma(\mathrm{Sm}_S) \arrow[r, "\text{add based points}"] \arrow[d, "\text{motivic localization}"'] & \mathcal{P}_\Sigma(\mathrm{Sm}_{S+}) \simeq \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet \arrow[d, "\text{motivic localization}"] \\
\mathcal{H}(S) \arrow[r, "\text{add based points}"] & \mathcal{H}_\bullet(S) \arrow[d, "\mathbb{P}^1\text{-stabilization } \Sigma^\infty_{\mathbb{P}^1}"] \\
 & \mathcal{SH}(S) := \mathcal{H}_\bullet(S)[\mathbb{P}_S^{-1}]
\end{tikzcd}$$

Figure 2. The construction of stable motivic homotopy category

Remark 2.2. The stable ∞-category $\mathcal{SH}(S)$ is the (homotopy) limit of the tower

$$\cdots \to \mathcal{H}_\bullet(S) \xrightarrow{\Omega^{\mathbb{A}^1}} \mathcal{H}_\bullet(S) \xrightarrow{\Omega^{\mathbb{A}^1}} \mathcal{H}_\bullet(S).$$

The construction of $p_\otimes$ is based on this process:

- **unbased presheaves level:** Write down $p_* \colon \mathcal{P}_\Sigma(\mathrm{Sm}_T) \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)$;
- **based presheaves level:** Write down $p_\otimes \colon \mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$
- **motivic spaces level:** $p_\otimes$ preserves motivic equivalences for good morphisms
- **motivic spectra level:** $p_\otimes$ preserves motivic Thom spaces for some family of morphisms.

2.1. **Multiplicative transfer on unbased presheaves.** In this subsection, we discuss the notion of Weil restrictions, which will be used considering the stabilizations. Basically, we expect that if $V \to X$ is an algebraic vector bundle in Sm_T, and $p\colon T \to S$ is a good morphism (we will see what "good morpihsm" means), then $p_\otimes(V) \to p_\otimes(X)$ is also a vector bundle. More precisely, both $p_\otimes(V)$ and $p_\otimes(X)$ are represented by smooth S-schemes.

Definition 2.3. Let $f\colon Q \to R$ be a morphism of scheme. The **pushforward**

$$f_*\colon \mathcal{P}(\mathrm{Sch}_Q) \to \mathcal{P}(\mathrm{Sch}_R)$$

is defined by

$$f_*F(X) := F(X \times_R Q),\ X \in \mathrm{Sm}_R,\ F \in \mathcal{P}(\mathrm{Sm}_Q)$$

If the F is represented by $X \in \mathrm{Sch}_Q$ and f_*X is representable in Sch_R, then the representative R-scheme is denoted by $\mathrm{R}_f X$ called **Weil restriction** of X along f.

Example 2.4. Let L/K be a finite Galois extension with Galois group G, let $V \in \mathrm{Sm}_L$. The Weil restriction of X along L/K should be a pair $(\tilde{X}, \varphi)$ where

- $\tilde{X} \in \mathrm{Sm}_K$;
- $\varphi\colon \tilde{X}_L \to X$ a morphism of smooth schemes.

such that for any smooth scheme T over K and $f: T_L \to X$, there exists a unique smooth morphism $\tilde{f}\colon T \to X$ such that $\varphi \circ (\tilde{f})_L = f$, namely

$$\begin{tikzcd}
 & \tilde{X} \arrow[dr, Rightarrow] & \\
T \arrow[ur, dashed, "\exists!\,\tilde{f}"] \arrow[dr, Rightarrow] & & \tilde{X}_L \arrow[d, "\varphi"] \\
 & T_L \arrow[ur, dashed, "(\tilde{f})_L"] \arrow[r, "f"'] & X
\end{tikzcd}$$

Now we let X be a smooth quasi-projective scheme over L. Let

$$\mathrm{Nm}_G(X) := \prod_{\sigma \in G} \sigma^* X$$

Then, for $\sigma \in \mathrm{Gal}(L/K)$, there exists an isomorphism $\varphi_\sigma\colon \mathrm{Nm}_G(X) \simeq \mathrm{Nm}_G(X)$ over $\mathrm{Spec}(L)$ given by $\mathrm{id} \times \sigma^*$, such that $(\mathrm{Nm}_G(X), \{\varphi_\sigma\}_{\sigma \in \mathrm{Gal}(L/K)})$ is an effective descent datum, see Appendix A, Theorem A.12. Then we let $\tilde{X} \in \mathrm{Sm}_K$ such that

$$(\mathrm{Nm}_G(X), \{\varphi_\sigma\}_{\sigma \in \mathrm{Gal}(L/K)}) \sim \tilde{X}.$$

In particular, $\tilde{X}_L \cong \mathrm{Nm}_G(X)$, according to the definition of effective descent data, and

$$(\tilde{f})_L = \prod_\sigma \sigma^* f$$

which corresponds to a morphism $\tilde{f}\colon T \to \tilde{X}$ uniquely[1]

Example 2.5. Given a finite field extension L/K of order d. If we specify a K-basis $\{e_1, \cdots, e_d\}$ of L, then for an affine L-space $\mathbb{A}^n_L = \mathrm{Spec} L[x_1, \cdots, x_n]$, the Weil restriction $\mathrm{R}_{L/K}(\mathbb{A}^n_L)$ is given by

$$\mathrm{Spec} K[y_{ij}] = \mathbb{A}^{nd}_K$$

Similarly, we have an analogous result for projective spaces.

[1]It is because the essential image of the fully faithful functor from Sm_K to the category of Galois descent data along L/K is exactly the category of effective Galois descent data.

Proposition 2.6. If V admits has the Weil restriction, then either a Zariski-open L-subscheme of V or a closed L-subscheme has its Weil restriction.

Proof. See [Wei82, Section 1.3]. ☐

Theorem 2.7. Let $p\colon T \to S$ be a finite locally free morphism between schemes, and X a quasi-projective T-scheme. Then the Weil restriction $\mathrm{R}_p X$ exists and is quasi-projective over S.

Proof. See [BLR90, Section 7.6, Theorem 4]. ☐

Remark 2.8. We need to require that the morphism should be finite locally free, because we need to basis affine-locally, as we do in Example 2.5.

Proposition 2.9. Let T be an arbitrary scheme and $X \in \mathrm{Sm}_T$. Suppose there is a finite locally free morphism $p\colon T \to S$. $\mathrm{R}_p X$ is smooth over S whenever the Weil restriction exists.

2.2. **Multiplicative transfer on based presheaves.** Extend p_* from non-pointed case to pointed case at the level of presheaves. More specifically, we need to extend it to a functor $\mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \xrightarrow{p_\otimes} \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$ such that the requirements are satisfied.

Note that $\mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet$ is generated under sifted colimits by objects of the form X_+. However, there is an obstruction on the way to the based cases: some maps in $\mathcal{P}_\Sigma(\mathrm{Sm})_T$ may not come from the functor $X \mapsto X_+$. For example, $f\colon (X \sqcup Y)_+ \to X_+$ that collapse Y to the base point cannot come from any map $X \sqcup Y \to X$. Therefore, the key point is how we define such $p_*(X \sqcup Y)_+ \to p_*(X)_+$.

Here we specialize the case to the case where $X, Y \in \mathrm{Sm}_T$. Then for any $U \in \mathrm{Sm}_S$, we decode the items:

(1) $p_*(X \sqcup Y)_+(U) = \mathrm{Sm}_T(U \times_S T, X \sqcup Y)_+$;
(2) $p_*(X)_+(U) = \mathrm{Sm}_T(U \times_S T, X)_+$;

For any $s : U \times_S T \to X \sqcup Y$, how should we define $p_\otimes(f)(s) \in p_*(X)_+$? Notice that we should collapse the part $s|_{s^{-1}(Y)}\colon s^{-1}(Y) \to Y$ according to the definition of f.

$$
\begin{tikzcd}
U \times_S T \arrow[r, "s", no head] & {} \arrow[r] \arrow[d, "\text{collapse the ``cross terms''}"] & X \sqcup Y \\
U \times_S T - s^{-1}(Y) \arrow[r, no head] & {} \arrow[r] & X
\end{tikzcd}
$$

However, the bottom arrow is not an element in $p_*(X)(U)$ evidently, which is regarded as a "cross term" in $p_*(X)(U)$. To make it more clear, we need to separate $s|_{s^{-1}(Y)} : s^{-1}(Y) \to Y$ from $s\colon U \times_S T \to X \sqcup Y$ in $p_*(X \sqcup Y)_+(U)$ by decomposing the presheaf $p_*(X \sqcup Y)_+$.

Definition 2.10. A **relatively representable** morphism is a morphism $Y \to X$ in $\mathcal{P}(\mathrm{Sm}_T)$ is such that the presheaf $V \times_X Y$ is representable whenever $V \to X$ for some $V \in \mathrm{Sm}_T$.

Lemma 2.11. For any coproduct decomposition $X = X_1 \sqcup X_2$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, the natural inclusion $X_1 \hookrightarrow X$ is relative representable.

Proof. Let $j_i : X_i \to X$ be the natural inclusion for each $i = 1, 2$. For any $Y \in \mathrm{Sm}_S$, by the universality of colimits, we have $Y = Y_1 \sqcup Y_2$, where

$$Y_i = Y \times_X X_i \text{ for } i = 1, 2$$

Recall that $\mathscr{O}(Y) = \mathrm{Hom}_{\mathrm{Sm}_S}(Y, \mathbb{A}^1)$. Then we can decompose the ring of functions of Y into

$$\mathrm{Hom}_{\mathrm{Sm}_S}(Y, \mathbb{A}^1) = \mathrm{Hom}_{\mathrm{Sm}_S}(Y_1 \sqcup Y_2, \mathbb{A}^1) = \mathrm{Hom}_{\mathrm{Sm}_S}(Y_1, \mathbb{A}^1) \times \mathrm{Hom}_{\mathrm{Sm}_S}(Y_2, \mathbb{A}^1)$$

By reducing the case to affine cases, we can decompose Y into two clopen subsets that represents Y_1 and Y_2 respectively. ☐

Construction 2.12. Let $Y_1, \dots, Y_k \to X$ be relatively representable morphisms. For $U \in \mathrm{Sm}_S$, let

$$p_*(X|Y_1, \dots, Y_k)(U) := \{s \colon U \times_S T \to X \mid s^{-1}(Y_i) \to U \text{ is surjective for all } i\}$$

where $s^{-1}(Y_i) \to U$ is given by the middle vertical composition of arrows in the following diagram.

$$\begin{tikzcd} & s^{-1}(Y_i) \arrow[r] \arrow[d] & Y_i \arrow[d] \\ T \arrow[d, "p"] & U \times_S T \arrow[l] \arrow[r, "s"] \arrow[d, "p_U"] & X \\ S & U \arrow[l] & \end{tikzcd}$$

Note that $p_*(X|Y_1, \dots, Y_k)$ is a subpresheaf of $p^*(X)$. If $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$,represents $p_*(X|Y_1, \dots, Y_k)$.

Lemma 2.13. Given a universally clopen morphism $p \colon T \to S$, $Y \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, with relatively representable morphisms $Z_1, \dots, Z_k \to Y$, for every coproduct decomposition $Y \simeq Y' \sqcup Y''$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_T)$, there is a decomposition

$$p_*(Y|Z_1, \dots, Z_k) \simeq p_*(Y'|Z_1', \dots, Z_k') \sqcup p_*(Y|Y'', Z_1, \dots, Z_k)$$

in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, where $Z_i' = Z_i \times_Y Y'$.

Proof. First, we reduce the case to $k = 0$:

(1) Let $\phi \colon p_*(Y') \sqcup p_*(Y|Y'') \to p_*(Y)$ be the morphism induced by the inclusions.
(2) Note that $p_*(Y'|Z_1', \dots, Z_k') = p_*(X') \cap p_*(Y|Z_1, \dots, Z_k)$.
(3) Note that $p_*(Y|Y'', Z_1, \dots, Z_k) = p_*(Y|Y'') \cap p_*(Y|Z_1, \dots, Z_k)$.
(4) Consider the following cartesian square

$$\begin{tikzcd} p_*(Y'|Z_1', \dots, Z_k') \sqcup p_*(Y|Y'', Z_1, \dots, Z_k) \arrow[r] \arrow[d, hook] & p_*(Y|Z_1, \dots, Z_k) \arrow[d, hook] \\ p_*(Y') \sqcup p_*(Y|Y'') \arrow[r, "\phi"] & p_*(Y) \end{tikzcd}$$

(5) We just need to show ϕ is an equivalence.

Then we specialize to the case $k = 0$, and show ϕ is a monomorphism:

(1) $p_*(Y') \times_{p_*(Y)} p_*(Y|Y'')$ has no sections over nonempty schemes, because $Y' \cap Y'' = \emptyset$ in Y.
(2) Hence $p_*(Y') \times_{p_*(Y)} p_*(Y|Y'')$ is an initial object of $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$, which means that ϕ is an equivalence by the universality of colimits. In particular, ϕ is a monomorphism.

It remains to show that ϕ is objectwisely an effective epimorphism:

(1) Let $p_U \colon U \times_S T \to U$ be the morphism parallel to $p \colon T \to S$ in the evident cartesian square.
(2) Given $U \in \mathrm{Sm}_S$ and $s \in p_*(Y)(U)$, we will decompose U according to these data.
(3) Let $U' = \{y \in U \mid p_U^{-1}(x) \subset s^{-1}(Y')\}$.
(4) Let U'' be the complement of U' in U, and $U'' = p_U(s^{-1}(Y''))$, which is a clopen subset of U.
(5) The image of the restriction $s|_{U'} \colon U' \to Y$ is in Y', according to the construction. Hence $s|_{U'} \in p_*(Y')(U')$ and we have $U' \to p_*(Y')$.
(6) $s^{-1}(Y'') \to U''$ is surjective, according to the construction. Hence $s|_{U'} \in p_*(Y')(U')$ and we have $U'' \to p_*(Y|Y'')$.

(7) Combine these coproducts together to define a section

$$U = U' \sqcup U'' \to p_*(Y') \sqcup p_*(Y|Y'')$$

which is a preimage of s by ϕ_U.

□

Remark 2.14. The proof of the surjectivity is the essential part, where we notice that the decomposition

$$p_*(Y') \sqcup p_*(Y|Y'') \simeq p_*(Y)$$

essentially encodes the decomposition of each section $s\colon U \to p_*(Y)$

$$\begin{tikzcd}
& U \arrow[d, equal] & \\
U' \arrow[r] \arrow[d] & U' \sqcup U'' \arrow[d] & U'' \arrow[l] \arrow[d] \\
p_*(Y') \arrow[r] & p_*(Y') \sqcup p_*(Y|Y'') \arrow[d, "\phi"] & p_*(Y|Y'') \arrow[l] \\
& p_*(Y) &
\end{tikzcd}$$

We may conclude that $s\colon U \to p_*(Y)$ is in $p_*(Y|Y'')$ if and only if the corresponding map $U \times_S T \to X$ can be lift to $U \times_S T \to Y \to Y''$ along the inclusion $Y'' \hookrightarrow Y$. If we let $f\colon Y_+ \to Y'_+$ collapse Y'' to the base point, the right vertical arrow $U'' \to p_*(Y|Y'')$ can be interpreted as the "cross terms" that should collapse. Therefore, we can see how $p(Y|Y'')$ packs the "cross terms".

Example 2.15. Given a universally clopen morphism $p\colon E \to B$ and let $Y, Z \in \mathcal{P}_\Sigma(\mathrm{Sm}_E)$, we have the decomposition

$$p_*(Y \sqcup Z) \simeq p_*(\emptyset) \sqcup p_*(Y|Y) \sqcup p_*(Z|Z) \sqcup p_*(Y \sqcup Z|Y, Z)$$

Theorem 2.16. Given a universally clopen morphism $p : T \to S$, there is a unique symmetric monoidal functor

$$p_\otimes\colon \mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$$

such that

(1) sifted colimits are preserved by $p_\otimes$;

(2) there is a natural equivalence $p_\otimes(X_+) \simeq p_*(X)_+$ between symmetric monoidal functors;

(3) for every $g\colon Z_+ \to Y_+$ with $Y, Z \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, the map $p_\otimes(g)$ is the composite

$$p_*(Z)_+ \to p_*(g^{-1}(Y))_+ \xrightarrow{f} p_*(Y)_+$$

by collapsing the part $p_*(Z|Z \setminus f^{-1}(Y))$ to the base point.

2.3. **Multiplicative transfer on motivic spaces.**

Proposition 2.17. Given an integral morphism $p : T \to S$ of schemes, Nisnevich and motivic equivalences are preserved by the functor $\mathcal{P}_\Sigma(\mathrm{Sm}_T) \xrightarrow{p_*} \mathcal{P}_\Sigma(\mathrm{Sm}_S)$.

Here we need to require the morphism to be integral because integral morphisms are direct limits of finitely presented morphisms. In this way, we can reduce the case to finite morphisms and further to finite field extension stalkwisely.

Remark 2.18. The condition that p is "integral and universally open" contains the following two cases:

(1) p is finite locally free;
(2) $p\colon \mathrm{Spec}L \to \mathrm{Spec}K$ is induced by an algebraic field extension L/K.

Combine Theorem 2.16 and Proposition 2.17, we have the following theorem.

Theorem 2.19. Let $p\colon T \to S$ be an integral and universally open morphism. Then there is a symmetric monoidal functor called **multiplicative transfer**

$$p_\otimes\colon \mathcal{H}_\bullet(T) \to \mathcal{H}_\bullet(S)$$

such that

(1) Sifted colimits is preserved by $p_\otimes$.
(2) p_* is extended by $p_\otimes$ i.e. $p_\otimes(Y_+) \simeq (p_*Y)_+$ for $Y \in \mathrm{Sm}_S$.
(3) If $p\colon S \times \{\underline{n}\} \to S$ is the trivial projection (we may also write it into a fold map $S^{\sqcup n} \to S$), then $p_\otimes$ is the n-fold smash product.

2.4. **Multiplicative transfer on motivic spectra.** In this subsection, we first recall the universal property of stable motivic homotopy categories. Then we use the universal property to extend multiplicative transfers from unstable motivic homotopy categories to stable motivic homotopy category.

Definition 2.20. Let X be a scheme and $V \to X$ be an algebraic vector bundle. The **motivic Thom space** associated to V is defined by

$$\mathrm{Th}(V) := V/(V \setminus 0) \simeq \mathbb{P}(V \oplus \mathbb{A}^1_X)/\mathbb{P}(V) \in \mathcal{H}_\bullet(S).$$

Example 2.21. Given a scheme S, $\mathrm{Th}(\mathbb{A}^1_S) = \mathbb{P}^1_S$.

Lemma 2.22. For any vector bundle $V \to S$, $\mathrm{Th}(V)$ is an invertible object in $\mathcal{SH}(S)$.

Theorem 2.23. Let $\mathcal{C}$ be a compact generated symmetric monoidal ∞-category whose tensor product preserves compact objects and colimits in each variable. Let $X \in \mathcal{C}$ such that the cyclic permutation on $X^{\otimes n}$ is homotopical to the identity for some $n \geq 2$. Let $\mathcal{K}$ be a collection of simplicial sets containing filtered simplicial sets. Let $\mathcal{D}$ be a symmetric monoidal ∞-category admitting $\mathcal{K}$-indexed colimits and whose tensor product preserves $\mathcal{K}$-indexed colimits in each variable. Then the localization

$$\Sigma^\infty_X\colon \mathcal{C} \to \mathcal{C}[X^{-1}]$$

induced a fully faithful embedding

$$\mathcal{F}\mathrm{un}^{\otimes,\mathcal{K}}(\mathcal{C}[X^{-1}], \mathcal{D}) \hookrightarrow \mathcal{F}\mathrm{un}^{\otimes,\mathcal{K}}(\mathcal{C}, \mathcal{D}),$$

whose essentially image consists of functors F such that $F(X)$ is invertible.

According to Theorem 2.23 and $\mathcal{SH}(S) = \mathcal{H}_\bullet(S)[\mathbb{P}^{-1}_S]$, we just need to show that $p_\otimes(\mathbb{P}^{-1}_T)$ is invertible. More precisely, the strategy consists of two steps:

(1) Let $V \to T$ be a vector bundle, show that $p_\otimes(V) \to S$ is a vector bundle.
(2) Show what $p_\otimes(X/Z)$ is and prove that $p_\otimes$ sends Thom spaces to Thom spaces.

Lemma 2.24. Let $p\colon T \to S$ be a finite étale morphism and $V \to T$ be a vector bundle. The Weil restriction $\mathrm{R}_pV \to S$ has a canonical vector bundle.

If $p\colon T \to S$ is a finite étale and $V \to T$ is a vector bundle, its Weil restriction $\mathrm{R}_pV \to S$ has a canonical structure of vector bundle (stalkwisely, it is Example 2.5).

Definition 2.25. Let $p\colon T \to S$ be a morphism of schemes, let $X \in \mathcal{P}(\mathrm{Sm}_T)$, and let $Y \subset X$ be a subsheaf. For $U \in \mathrm{Sm}_S$, let

$$p_*(X||Y)(U) = \{s\colon U \times_S T \to X \mid s \text{ sends a clopen subset covering } U \text{ to } Y\}.$$

Note that $p_*(X||Y) \subset p_*(X)$, and it is in $\mathcal{P}_\Sigma$ whenever X and Y are.

Proposition 2.26. Given a universally clopen morphism $p : T \to S$, $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, and a subpresheaf $Y \subset X$ in $\mathcal{P}_\Sigma(\mathrm{Sm}_T)$, there is a natural equivalence

$$p_\otimes(X/Y) \simeq p_*(X)/p_*(X||Y)$$

in $\mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet$.

Proposition 2.27. Let $p : T \to S$ be an integral universally open morphism, let $X \in \mathcal{P}_\Sigma(\mathrm{Sm}_T)$, and let $Y \subset$ be an open subsheaf. Then there is a natural equivalence

$$p_\otimes(X/Y) \simeq p_*(X)/p_*(X|Y)$$

in $\mathcal{S}\mathrm{hv}_{nis}(\mathrm{Sm}_S)_\bullet$.

Proof. See [BH21, Corollary 3.11]. □

Proposition 2.28. Given a finite étale morphism, $p\colon T \to S$, $X \in \mathrm{Sm}_T$, and a closed subscheme $Z \subset X$, if the Weil restriction $\mathrm{R}_p X$ exists, then

$$p_\otimes\left(\frac{X}{X \setminus Z}\right) \simeq \frac{\mathrm{R}_p X}{\mathrm{R}_p X \setminus \mathrm{R}_p Z}$$

Proposition 2.29. Given a finite étale morphism $p : T \to S$, a vector bundle V over T, we have $p_\otimes(S^V) \simeq S^{\mathrm{R}_p V}$ in $\mathcal{H}_\bullet(S)$.

Proposition 2.30. Given a finite étale morphism $p\colon T \to S$, the functor $\Sigma_\infty p_\otimes\colon \mathcal{H}_\bullet(T) \to \mathcal{SH}(S)$ has a unique symmetric monoidal extension

$$p_\otimes\colon \mathcal{SH}(T) \to \mathcal{SH}(S)$$

preserving sifted colimits.

Remark 2.31. Let $p\colon T \to S$ be finite étale morphism and let $E \in \mathcal{SH}(T)$. Then we have

$$p_\otimes(E) \simeq \mathrm{colim}_n \Sigma^{-\mathrm{R}_p \mathbb{A}^n} \Sigma^\infty p_\otimes(E_n)$$

where E_n is the nth space of E and $E \simeq \mathrm{colim}_n \Sigma^{-\mathbb{A}^n} \Sigma^\infty E_n$.

3. Properties and coherence of norms

In this section, we mainly introduce how the multiplicative norm functors interact with other operations coherently.

Proposition 3.1 (Composition)**.** Given two universally clopen morphisms $f\colon R \to T$ and $g\colon T \to S$, there is a symmetric monoidal natural equivalence

$$(gf)_\otimes \simeq g_\otimes f_\otimes \colon \mathcal{P}_\Sigma(\mathrm{Sm}_R)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_T)_\bullet \to \mathcal{P}_\Sigma(\mathrm{Sm}_S)_\bullet.$$

Hence, the same result holds in $\mathcal{H}_\bullet$ (resp. in $\mathcal{SH}$) if f and g are integral and universally open (resp. are finite étale).

Proposition 3.2 (Base change)**.** Given a pull-back square of schemes as follows

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d, "q"'] & T \arrow[d, "p"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

where p is universally clopen. Let $\mathcal{C} \subset \mathrm{Sm}_T$ be a full subcategory and let $X \in \mathcal{P}_\Sigma(\mathcal{C})_\bullet$. Suppose either of the following assertions is true

(1) f is smooth;

(2) the Weil restriction $\mathrm{R}_p U$ is a smooth S-scheme for every $U \in \mathcal{C}$,

Then there exists a natural equivalence $\mathrm{Ex}^*_\otimes \colon f^* p_\otimes(X) \to q_\otimes g^*(X)$. In particular, if p is finite étale (resp. finite locally free), then there is an equivalence $\mathrm{Ex}^*_\otimes \colon f^* p_\otimes \to q_\otimes g^*$ equivalence in $\mathcal{SH}$ (resp. in $\mathcal{H}_\bullet$).

Remark 3.3. By taking adjunction for $\mathrm{Ex}^*_\otimes \colon f^* p_\otimes \to q_\otimes g^*$, we have

$$\mathrm{Ex}_{\otimes *} \colon p_\otimes g_* \to f_* q_\otimes$$

If f is smooth, we also have

$$\mathrm{Ex}_{\# \otimes} \colon f_\# q_\otimes \to p_\otimes g_\#$$

Given a finite locally free morphism $p\colon T \to S$ and a quasi-projective morphism $h\colon Q \to T$, we have the diagram

$$\begin{tikzcd} Q \arrow[drr, "h"'] & \mathrm{R}_p Q \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & \mathrm{R}_p Q \arrow[d, "f"] \\ & T \arrow[r, "p"] & S \end{tikzcd}$$

where e is the counit of the adjunction (p^*, p_*), q and g are the canonical projections, and $f = \mathrm{R}_p(h)$. Then we define

$$\mathrm{Dis}_{\# *} \colon f_\# q_* e^* \xrightarrow{\mathrm{Ex}_{\# *}} p_* g_\# e^* \xrightarrow{\epsilon} p_* h_\# \colon \mathrm{QP}_U \to \mathrm{QP}_S$$

Furthermore, we consider

$$\mathrm{Dis}_{\# \otimes} \colon f_\# q_\otimes e^* \xrightarrow{\mathrm{Ex}_{\# \otimes}} p_\otimes g_\# e^* \xrightarrow{p}_\otimes h_\#,$$

$$\mathrm{Dis}_{\otimes *} \colon p_\otimes h_* \xrightarrow{\eta} p_\otimes g_* e^* \xrightarrow{\mathrm{Ex}_{\otimes *}} f_* q_\otimes e^*$$

3.1. **Categorical encapsulation of motivic norms and their coherence.** To organize these properties and their coherence more efficiently, we introduce the notion of spans.

Definition 3.4. Given a category $\mathcal{C}$ with two classes of morphisms L and R such that

- they all contains equivalences,
- the pull-back of any arrow in L (resp. in R) along any arrow in R (resp. in L) is still in L (resp. in R),
- they are closed under compositions,

we construct a new ∞-category $\mathrm{Span}(\mathcal{C}, \mathrm{L}, \mathrm{R})$ whose objects are objects in $\mathcal{C}$ and morphisms are of the form

$$\bullet \xleftarrow{f} \bullet \xrightarrow{g} \bullet$$

where $f \in \mathrm{L}$ and $g \in \mathrm{R}$. The composition is given by pull-back.

In this subsection, we will construct the functor

$$\mathcal{SH}^{\otimes} \colon \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty),\ S \mapsto \mathcal{SH}(S), (U \xleftarrow{f} T \xrightarrow{p} S) \mapsto p_\otimes f^*$$

such that

- If p, q are composable finite étale maps, then $(q \circ p)_\otimes \simeq q_\otimes \circ p_\otimes$.
- given a cartesian square

$$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "q"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "f"'] & \bullet \end{tikzcd}$$

  with p finite étale , $f^* \circ p_\otimes \simeq q_\otimes \circ g^*$
- coherence of the above equivalences.

The strategy to construct this functor follows the diagram 2 basically, but we still need some modification: we need to replace Sm_S by SmQP_S. There are two reasons for this modification:

- In the case of smooth quasi-projective schemes, the existence of the Weil restriction is guaranteed, see Proposition Theorem 2.7. In this way, we can restrict the case to $p_* : \mathrm{SmQP}_T \to \mathrm{SmQP}_S$, and simplify the whole machinery.
- There is no harm, since $\mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{SmQP}_S) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$, which means that we can still get $\mathcal{H}(S)$ from $\mathcal{P}_\Sigma(\mathrm{Sm}_S)$ by motivic localization. In other words, $\mathcal{H}(S)$ is generated by SmQP_S under sifted colimits.

Based on these observations, the construction follows the next process:

$$\mathrm{SmQP}_{S+} \rightsquigarrow \mathcal{P}_\Sigma(\mathrm{SmQP}_S)_\bullet \rightsquigarrow \mathcal{H}_\bullet(S) \rightsquigarrow \mathcal{SH}(S)$$

More precisely,

(1) The assignment $S \mapsto \mathrm{SmQP}_{S+}$ forms a functor $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_1)$;

(2) By sifted cocompletion, we can extend the previous one to $S \mapsto \mathcal{P}_\Sigma(\mathrm{SmQP}_S)_\bullet$, which forms a functor $\mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{sift}})$;

(3) Show the assignment preserves motivic equivalences, so we can pass to

$$\mathcal{H}(-)_\bullet^{\otimes} \colon \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlgCAlg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{sift}}),\ S \mapsto \mathcal{H}(S)_\bullet.$$

(4) Show the assignment preserves Thom spaces so that we can pass to

$$\mathcal{SH}(-)_\bullet^{\otimes} \colon \mathrm{Span}(\mathrm{Sch}, \mathrm{all}, \mathrm{f\acute{e}t}) \to \mathrm{CAlgCAlg}(\mathcal{C}\mathrm{at}_\infty^{\mathrm{sift}}),\ S \mapsto \mathcal{SH}(S).$$

The non-trivial ones are the last two steps. Bachmann and Hoyois prove them in a very smart and elegant way in [BH21, §6.1]. Their proof is basically using categorical machinery.

Beyond the construction $\mathcal{SH}^{\otimes}$, we can generalized $\mathcal{SH}^{\otimes}$ to define a notion describing a family of symmetric monoidal ∞-categories parameterized by schemes with multiplicative transfers associated to finite étale morphisms.

Let S be a scheme. We write $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$ if $\mathcal{C}$ is a full subcategory of Sch_S that contains S and is closed under finite coproducts and finite étale extensions. We denote fét the class of finite étale morphisms.

Definition 3.5 (Normed ∞-category)**.** Let S be a scheme and $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$. A *normed ∞-category* over $\mathcal{C}$ is a functor

$$\mathcal{A}\colon \mathrm{Span}(\mathcal{C}, \mathrm{all}, \text{fét}) \to \mathcal{C}\mathrm{at}_\infty,\ \ (X \xleftarrow{f} Y \xrightarrow{p} Z) \mapsto p_\otimes f^*,$$

preserving finite products. $\mathcal{A}$ is said to be *presentably normed* if:

(1) $\mathcal{A}(X)$ is presentable for every $X \in \mathcal{C}$;

(2) $h^*\colon \mathcal{A}(X) \to \mathcal{A}(Y)$ has a left adjoint $h_\#$ for every finite étale morphism $h\colon Y \to X$;

(3) $f^*\colon \mathcal{A}(X) \to \mathcal{A}(Y)$ preserves colimits for every morphism $f\colon Y \to X$;

(4) for every pull-back square

$$\begin{tikzcd} Y' \arrow[r, "g"] \arrow[d, "h'"'] & Y \arrow[d, "h"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

where h is a finite étale morphism, there an equivalence

$$\mathrm{Ex}^*_\#\colon h'_\# g^* \to f^* h_\# \colon \mathcal{A}(Y) \to \mathcal{A}(X')$$

as an exchange transformation;

(5) $p_\otimes\colon \mathcal{A}(Y) \to \mathcal{A}(Z)$ preserves sifted colimits for every finite étale morphism $p\colon Y \to Z$;

(6) for every diagram

$$\begin{tikzcd} U & \mathrm{R}_p U \times_S T \arrow[l, "e"'] \arrow[r, "q"] \arrow[d, "g"] & \mathrm{R}_p U \arrow[d, "f"] \\ & T \arrow[r, "p"] \arrow[from=1-1, "h"'] & S \end{tikzcd}$$

where p and h are finite étale morphisms, there exists an equivalence

$$\mathrm{Dis}_{\#\otimes}\colon f_\# q_\otimes e^* \to p_\otimes h_\#$$

as the distributivity transformation .

3.2. **The category of normed motivic spectra.** Recall that if $\mathcal{A} : \mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ is a functor classifying a cocartesian fibration $p : \mathcal{E} \to \mathcal{C}$, a *section* of $\mathcal{A}$ is a section $s : \mathcal{C} \to \mathcal{E}$ of p. More specifically, for any $c \in \mathcal{C}$, $s(c)$ is an object in $\mathcal{A}(c)$. We write

$$\int \mathcal{A} = \mathcal{E} \text{ and } \mathrm{Sect}(\mathcal{A}) = \mathcal{F}\mathrm{un}_{\mathcal{C}}(\mathcal{C}, \mathcal{E})$$

Definition 3.6. Let $S \in \mathrm{Sch}$ and $\mathcal{C} \subset_{fet} \mathrm{Sch}_S$. A *normed spectrum over* $\mathcal{C}$ is a section of $\mathcal{SH}^{\otimes}$ over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \text{fét})$ that is cocartesian over $\mathcal{C}^{\mathrm{op}}$. An *incoherent normed spectrum over* $\mathcal{C}$ is a section of $h\mathcal{SH}^{\otimes}$ over $\mathrm{Span}(\mathcal{C}, \mathrm{all}, \text{fét})$ that is cocartesian over $\mathcal{C}^{\mathrm{op}}$.

The full subcategory of normed spectra over $\mathcal{C}$ is denoted by $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \subset \mathrm{Sect}(\mathcal{SH}^{\otimes} \mid \mathrm{Span}(\mathcal{C}, \mathrm{all}, \text{fét}))$. The frequent choices of $\mathcal{C}$ are Sm_S, Sch_S and FEt_S. For convenience, we write $\mathrm{NAlg}_{\mathrm{Sm}}(\mathcal{SH}(S))$ instead of $\mathrm{NAlg}_{\mathrm{Sm}_S}(\mathcal{SH})$.

Roughly speaking, a normed spectrum E over $\mathcal{C}$ is to assign $E_X \in \mathcal{SH}(X)$ for any $X \in \mathcal{C}$ and $p_\otimes f^* E_X \to E_Z$ in $\mathcal{SH}(Z)$ for any span $X \xleftarrow{f} Y \xrightarrow{p} Z$. Note that by full-back, we have that $f^* E_Y = E_X$ naturally. Therefore, the extra data for an (incoherent) normed structure is a spectrum $E \in \mathrm{Sch}(S)$ equipped with a parametrized multiplicative transfer $\mu_p : p_\otimes E_V \to E_U$ for any finite étale morphism $p \colon V \to U$ in $\mathcal{C}$ such that the following coherence conditions are satisfied.

Condition 3.7 (Coherence conditions for incoherent normed spectra)**.**

(1) μ_p is an equivalence when p is the identity;

(2) The square with two arbitrary composable finite étale morphisms $q \colon W \to V$ and $p \colon V \to U$ in $\mathcal{C}$

$$\begin{tikzcd} p_\otimes q_\otimes E_W \arrow[r, "p_\otimes \mu_q"] \arrow[d, "\simeq"'] & p_\otimes E_V \arrow[d, "\mu_p"] \\ (pq)_\otimes E_W \arrow[r, "\mu_{pq}"'] & E_U \end{tikzcd}$$

commutes up to homotopy.

(3) for every pull-back square

$$\begin{tikzcd} V' \arrow[r, "g"] \arrow[d, "q"'] & V \arrow[d, "p"] \\ U' \arrow[r, "f"'] & U \end{tikzcd}$$

in $\mathcal{C}$ where p is a finite étale morphism, the following diagram

$$\begin{tikzcd} f^* p_\otimes E_V \arrow[r, "f^*\mu_p"] \arrow[d, "\simeq"'] & f^* E_U \arrow[dd, "\simeq"] \\ q_\otimes g^* E_V \arrow[d, "\simeq"'] & \\ q_\otimes E_{V'} \arrow[r, "\mu_q"'] & E_{U'} \end{tikzcd}$$

commutes up to homotopy.

In particular, these coherence conditions imply that $\mu_p \colon p_\otimes E_V \to E_U$ is homotopically equivariant for the action of $\mathrm{Aut}(V/U)$ on $p_\otimes E_V$. Thus we have

$$\mu_p \colon (p_\otimes E_V)_{h\mathrm{Aut}(V/U)} \to E_U$$

Basically, the multiplicative coherence data for a normed spectrum over $\mathcal{C} \subset_{\text{fét}} \mathrm{Sm}_S$ is parametrized by $\mathcal{C} \cap \mathrm{FEt}_S$.

Proposition 3.8. Suppose S is a scheme and $\mathcal{C} \subset_{\text{fét}} \mathrm{Sch}_S$.

(1) The ∞-category $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathcal{SH}(S)$ admits all finite limits and colimits. If $\mathcal{C}$ is a small ∞-category, then $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$ is presentable.

(2) The forgetful functor $\mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH}) \to \mathcal{SH}(S)$ is conservative and preserves sifted colimits and finite limits. If $\mathcal{C} \subset \mathrm{Sm}_S$, it preserves limits and hence is both monadic.

Proof. See [BH21, Proposition 7.6]. □

Remark 3.9. The forgetful functor $\mathrm{NAlg}(\mathcal{SH}) \to \mathcal{SH}(S)$ has a left adjoint $\mathrm{NSym}_{\mathcal{C}} : \mathcal{SH}(S) \to \mathrm{NAlg}_{\mathcal{C}}(\mathcal{SH})$. When $\mathcal{C} = \mathrm{Sm}_S$ or $\mathcal{C} = \mathrm{FEt}_S$, we have that

$$\mathrm{NSym}_{\mathcal{C}}(E) = \operatorname*{colim}_{\substack{f: X \to S \\ p: Y \to X}} f_\# p_\otimes (E_Y)$$

where the indexing ∞-category is the source of the cartesian fibration classified by $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, $X \mapsto \mathrm{FEt}_X^{\simeq}$. Therefore the motivic norm structure on a spectrum $E \in \mathcal{SH}(S)$ can be exhibited as

$$\mathrm{NAlg}_{\mathcal{C}}(E) \to E$$

The monadic argument can be found in [BH21, Section 7.1,16.4]. Motivated by this, Bachmann, Elmanto and Heller define the notion of motivic colimits [BEH21].

4. Norms in equivariant homotopy theory

4.1. **Unstable G-equivariant homotopy theory.** Let $f : H \to K$ be a group homomorphism. Then we have

$$\begin{array}{rccc} f_! : & \mathcal{S}\mathrm{pc}_G & \longrightarrow & \mathcal{S}\mathrm{pc}_K \\ & X & \longmapsto & K \times_H X \end{array}$$

$$\begin{array}{rccc} f_* : & \mathcal{S}\mathrm{pc}_G & \longrightarrow & \mathcal{S}\mathrm{pc}_K \\ & X & \longmapsto & \mathrm{Map}(K, X)^H \end{array}$$

and they forms adjoint functors

$$\begin{tikzcd} \mathcal{S}\mathrm{pc}_H \arrow[r, bend left, "f_!"] & \mathcal{S}\mathrm{pc}_K \arrow[l, bend left, "f^*"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{S}\mathrm{pc}_H \arrow[r, bend left, "f^*"] & \mathcal{S}\mathrm{pc}_K \arrow[l, bend left, "f_*"] \end{tikzcd}$$

where f^* is the pull-back of the group action:

$$\mathcal{F}\mathrm{un}(BK, \mathcal{S}\mathrm{pc}) \xrightarrow{\circ Bf} \mathcal{F}\mathrm{un}(BH, \mathcal{S}\mathrm{pc}).$$

We may called $f_!$ the **induction functor along** f and $f_!$ the **coinduction functor along** f.

4.2. **Set-up for equivariant stable homotopy theory.** Let G be a group, we let BG be the associated ∞-groupoid.

Let $\mathcal{S}\mathrm{pc}_G$ be the (1-)category of G-spaces and let $\mathcal{W}_1$ be the class of G-equivariant weak homotopy equivalence. Then we have

$$\mathcal{S}\mathrm{pc}_G[\mathcal{W}_1] \simeq \mathcal{F}\mathrm{un}(BG, \mathcal{S}\mathrm{pc})$$

where the LHS should be considered as the homotopy coherent nerve of the hammock localization of $\mathcal{S}\mathrm{pc}_G$ with respect to $\mathcal{W}_1$.

However, the problem is that since the categorical homotopy fixed point functor is not a homotopical functor, we may loss the information of the homotopy types of X^H for all non-trivial $H \subset G$. To fix this issue, we need to define a more suitable class of weak equivalences.

Definition 4.1. Let $\mathcal{W} \subset \mathcal{S}\mathrm{pc}_G$ be the class of morphisms $f : X \to Y$ such that $f^H : X^H \to Y^H$ is a G-weak homotopy equivalence for all $H \subset G$.

On the other hand, we need to consider how to modify BG, namely we need to find a better ∞-category to parameterize spaces for G-equivariant homotopy theory.

Definition 4.2. Let $\mathcal{O}_G$ be the category called **Elmendorf orbit category** with

- **objects:** G/H where H is a closed subgroup;
- **morphisms:** G-equivariant continuous functors. Specifically, there exists a $G/H_1 \to G/H_2$ if and only if $gH_1g^{-1} \subset H_2$ for some $g \in G$ (and the morphism is given by $[x] \mapsto [gxg^{-1}]$).

In particular, $\mathcal{S}\mathrm{pc}_G \simeq \mathcal{F}\mathrm{un}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$ as a homotopy cocompletion, since a G-space is a colimit of G-orbits. An alternative model for $\mathcal{S}\mathrm{pc}$ is $\mathcal{P}_\Sigma(\mathcal{F}in_G)$ where $\mathcal{F}in_G$ is the category of finite discrete G-sets.

When passing to stable world, we have several strategies

- **Borel G-spectra**: $\mathcal{S}\mathrm{p}^{hG} := \mathcal{F}\mathrm{un}(BG, \mathcal{S}\mathrm{p})$ is called the stable ∞-category of Borel G-spectral, which can be obtained by do S^1-stabilization on $\mathcal{F}\mathrm{un}(BG, \mathcal{S}\mathrm{pc})$;
- **Naive G-spectra**: $\mathcal{S}\mathrm{p}_G := \mathcal{F}\mathrm{un}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\mathrm{p})$ is the stable ∞-category of naive G-spectra, which can be obtained by S^1-stabilization on $\mathcal{F}\mathrm{un}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}\mathrm{pc})$.

- **Genuine G-spectra**: $\mathcal{S}\mathrm{p}^G := \mathcal{F}\mathrm{un}^{\oplus}(\mathrm{Span}^+(\mathcal{F}in_G), \mathcal{S}\mathrm{p})$ is the stable ∞-category of genuine G-spectra, where $\mathrm{Span}^+(\mathcal{F}in_G)$ is the **Burnside category** and a genuine G-spectra is also called a **spectral Mackey functor**.

Definition 4.3. Let $\mathrm{Span}(\mathcal{F}in_G) = \mathrm{Span}(\mathcal{F}in_G, \mathrm{all}, \mathrm{all})$. The space of the morphism space $\mathrm{Hom}_{\mathrm{Span}(\mathcal{F}in_G)}(X, Y)$ is a commutative monoid whose addition is given by coproducts

$$\begin{tikzcd} & X & & & X \\ (Z' \arrow[ur] \arrow[dr] & + & Z) \arrow[ul] \arrow[dl] \arrow[rr, equal] & & Z \coprod Z' \arrow[u] \arrow[d] \\ & Y & & & Y \end{tikzcd}$$

The **Burnside category** $\mathrm{Span}^+(\mathcal{F}in_G)$ is obtained from $\mathrm{Span}(\mathcal{F}in_G)$, which is called **effective Burnside category** by doing group completion on the mapping commutative monoid.

The extra data in the comparison between naive G-spectra and genuine G-spectra is about transfers in equivariant homotopy theory. We will see how these transfers are actually from transfers in representation theory by using the following definition of genuine G-spectra.

Definition 4.4. Let V be a G-representation and S^V be the representation sphere by one-point compactification. Then the stable ∞-category of genuine G-spectra is obtained by

$$\mathcal{S}\mathrm{p}^G = \mathcal{S}\mathrm{pc}_G[\{S^{-V}\}_{V \in \mathrm{Rep}(G)}]$$

namely it is given by the formal inversion with respect to all G-representation spheres.

Remark 4.5. If G is a finite group, then its regular representation $R(G)$ is the direct sum of all the irreducible G-representations. Then by the semi-simple property of G-modules, we have that

$$\mathcal{S}\mathrm{p}^G = \mathcal{S}\mathrm{pc}_G[S^{-R(G)}].$$

Note that $S^{V \oplus W} \cong S^V \wedge S^W$.

Now we recall some transfers in representation theory. Let $H \subset G$ be a subgroup, then we have a pair of adjoint functors

$$\begin{tikzcd} \mathrm{Rep}(H) \arrow[r, bend left, "\mathrm{Ind}_H^G"] & \mathrm{Rep}(G) \arrow[l, bend left, "\mathrm{Res}_H^G"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Rep}(G) \arrow[r, bend left, "\mathrm{Res}_H^G"] & \mathrm{Rep}(H) \arrow[l, bend left, "\mathrm{Coind}_H^G"] \end{tikzcd}$$

Then we may upgrade them to

$$\begin{tikzcd} \mathcal{S}\mathrm{pc}_H \arrow[r, bend left, "\mathrm{Ind}_H^G"] & \mathcal{S}\mathrm{pc}_G \arrow[l, bend left, "\mathrm{Res}_H^G"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{S}\mathrm{pc}_G \arrow[r, bend left, "\mathrm{Res}_H^G"] & \mathcal{S}\mathrm{pc}_H \arrow[l, bend left, "\mathrm{Coind}_H^G"] \end{tikzcd}$$

and the unstable norm is given by

$$\begin{tikzcd}
\mathcal{S}\mathrm{pc}_H \arrow[r, "\mathrm{Coind}_H^G"] \arrow[d, "(-)_+"'] & \mathcal{S}\mathrm{pc}_G \arrow[d, "(-)_+"] \\
\mathcal{S}\mathrm{pc}_{H\bullet} \arrow[r, "N_H^G"'] & \mathcal{S}\mathrm{pc}_{G\bullet}
\end{tikzcd}$$

By stabilization with respect to all representation spheres, we eventually have the equivariant stable norm functor N_H^G.

Appendix A. Appendix: Galois descent

A.1. **Galois descent for vector spaces.** Let L/K be a finite Galois extension with Galois group G. By tensor products, we have

$$\begin{array}{rccc} L\otimes_K - : & \mathrm{Vect}_K & \longrightarrow & \mathrm{Vect}_L \\ & V & \longmapsto & L\otimes_K V \end{array}$$

Question A.1. Given a L-vector space W, can we find a K-vector space V such that $W \cong L\otimes_K V$ as L-vector spaces?

Notice that there is a G-action on $L\otimes_K V$ by

$$g\cdot(a\otimes v) = (g\cdot a)\otimes v,\ \forall g\in G$$

and this G-action is compatible with its K-vector space structure, since g is a K-linear map. The following notion encapsulates these structures

Definition A.2. Let W be a L-vector space and $g\in G$. An additive function$\varphi\colon W\to W$ is said to be g-linear if

$$\varphi(aw) = g(a)\varphi(w),\ \forall v\in W,\, a\in L.$$

A G-structure on W is a set of functions $\varphi_g\colon W\to W$ such that φ_g is a g-linear function for all $g\in G$ and $\varphi_{g_1}\circ\varphi_{g_2} = \varphi_{g_1g_2}$. We may also say G acts semilineraly on V.

Example A.3. Given a K-vector space V, $L\otimes_K V$ has a standard G-structure

$$g\colon a\otimes v\mapsto g(a)\otimes v.$$

Proposition A.4. A L-vector space W is of the form $W\cong L\otimes_K V$ for some K-vector space V if and only if L has G-structure.

Roughly speaking, given a L-vector space W with G-structure, we have

$$L\otimes_K W^G \cong W$$

where W^G is the space of G-fixed points and it is a K-vector space in nature.

Let Mod_G be the category of L-vector space with G-structure. Proposition A.4 is saying that

$$\mathrm{Vect}_k \cong \mathrm{Mod}_G$$

A.2. **Descent data for quasi-coherent sheaves.**

Definition A.5. Let S be a scheme. Let $\{f_i\colon S_i\to S\}_{i\in I}$ be a family of morphisms. A **descent datum** $\{\mathcal{F}_i,\varphi_{ij})$ for quasi-coherent sheaves with respect to the given family is given by a quasi-coherent sheaf $\mathcal{F}_i$ on S_i for each $i\in I$, an isomorphism of $\mathcal{O}_{S_i\times S_j}$-modules:

$$\varphi_{ij}\colon \mathrm{pr}_0^*\mathcal{F}_i\to\mathrm{pr}_1^*\mathcal{F}_j$$

for each pair $(i,j)\in I^2$, where the morphisms are given by following cartesian square

$$\begin{tikzcd} S_i\times_S S_j \arrow[r, "\mathrm{pr}_0"] \arrow[d, "\mathrm{pr}_1"'] & S_i \arrow[d] \\ S_j \arrow[r] & S \end{tikzcd}$$

such that the cocycle condition for $\mathcal{O}_{S_i\times_S S_j\times_S S_k}$-modules

$$\begin{tikzcd} \mathrm{pr}_0^*\mathcal{F}_i \arrow[rr, "\mathrm{pr}_{02}^*\varphi_{ik}"] \arrow[dr, "\mathrm{pr}_{01}^*\varphi_{ij}"'] & & \mathrm{pr}_2^*\mathcal{F}_k \\ & \mathrm{pr}_1^*\mathcal{F}_j \arrow[ur, "\mathrm{pr}_{12}^*\varphi_{jk}"'] & \end{tikzcd}$$

given by

$$\begin{tikzcd}
 & & & & & S_i \times_S S_j \times_S S_k \arrow[dllll, "\mathrm{pr}_{01}"'] \arrow[d, "\mathrm{pr}_{02}"] \arrow[drrrr, "\mathrm{pr}_{12}"] & & & & & \\
 & S_i \times_S S_j \arrow[dl, "\mathrm{pr}_0"'] \arrow[dr, "\mathrm{pr}_1"] & & & & S_i \times_S S_k \arrow[dl, "\mathrm{pr}_0"'] \arrow[dr, "\mathrm{pr}_1"] & & & & S_j \times_S S_k \arrow[dl, "\mathrm{pr}_0"'] \arrow[dr, "\mathrm{pr}_1"] & \\
S_i \arrow[dr] & & S_j \arrow[dl] & & S_i \arrow[dr] & & S_k \arrow[dl] & & S_j \arrow[dr] & & S_k \arrow[dl] \\
 & S \arrow[rrrr, equal] & & & & S \arrow[rrrr, equal] & & & & S &
\end{tikzcd}$$

where the copies of S_i (resp. S_j, resp. S_k) are identified.

Definition A.6. Let S be a scheme and $\{f\colon S_i \to S\}_{i\in I}$ be a family of scheme morphisms. Let $\mathcal{F}$ be an $\mathcal{O}_S$-module.

- The identity morphism $\{\mathrm{id}\colon S \to S\}$ with its natural descent datum is called the **trivial descent**.
- The pull-back of the trivial descent along $\{f : S_i \to S\}$ i.e. $\{\mathcal{F}|_{S_i}, \mathrm{can}\}$ is the **canonical descent datum**;
- a descent data $\{\mathcal{F}_i, \varphi_{ij}\}$ on $\{f_i : S_i \to S\}$ is called an **effective descent datum** if it is isomorphic to a canonical descent datum for some $\mathcal{O}_S$-module $\mathcal{F}$.

Proposition A.7. Let $F := \{f_i\colon S_i \to S\}$ be a family of morphisms.

- If F is an open covering, then any descent datum is effective.
- If f_i is flat for any $i \in I$ and for any affine open set $V \subset S$, there exists a finite subset $J \subset I$ with affine open subsets $V_j \subset S_j$ for any j such that $\cup_{j\in J} f_j(V_j) = V$ (i.e. F is a **fpqc covering**), then any descent datum is effective.

A.3. **Galois descent data for schemes.** Let L/K be a finite Galois extension with Galois group G. In particular, the morphism $\{\mathrm{Spec}L \to \mathrm{Spec}K\}$ is a fpqc covering and an étale covering at the same time. Now we try to adapt Definition A.5 to the case of Galois extensions.

Lemma A.8. Let L_σ be a L-algebra given by $\sigma\colon L \to L \in G$. Then

$$L \otimes_K L \cong \prod_{\sigma\in G} L_\sigma,\ \ x \otimes y \mapsto (x\sigma(y))_\sigma$$

as L-algebras. Let G act on $\prod_{\sigma G} L_\sigma$ by $g \cdot (x_\sigma)_\sigma = (x_{\sigma g})_\sigma$ for any $g \in G$. Then we have $L \cong (\prod_{\sigma\in G} L_\sigma)^G$ as K-algebras. Similarly, $L \otimes_K \otimes_K L \cong \prod_{(\sigma,\tau)\in G\times G} L$ given by $a \otimes b \otimes c \mapsto (a\sigma(b)\tau(c))_{\sigma,\tau}$.

For any scheme X over K, we have $X_{L\otimes_K L} \cong \coprod_{\sigma\in G} X_{L_\sigma}$. For any $\sigma \in G$, we let $\sigma^*\colon \mathrm{Sch}_L \to \mathrm{Sch}_L$ be the functor induced by $\sigma^*\colon \mathrm{Spec}L \to \mathrm{Spec}L$ in the following way:

$$(Y \to \mathrm{Spec}L) \mapsto (Y \to \mathrm{Spec}L \xrightarrow{\sigma^*} \mathrm{Spec}L).$$

In particular, we let $T_\sigma\colon X_L \to X_L$ be $\mathrm{id} \times \sigma^*$ which is coincident with $\sigma^*|_{X_L} : X_L \to X_L$[2].

Then $\mathrm{pr}_0 : X_{L\otimes_K L} \to X_L$ corresponds to $\coprod_{\sigma\in G} X_L \xrightarrow{\coprod \mathrm{id}} X_L$ and $\mathrm{pr}_1 : X_{L\otimes_K L} \to X_L$ corresponds to $\coprod_{\sigma\in G} X_L \xrightarrow{\coprod \sigma^*} X_L$. Similarly, we can write down $\mathrm{pr}_{01}, \mathrm{pr}_{12}, \mathrm{pr}_{02}$ from $\coprod_{(\sigma,\tau)\in G\times G} X_L \to$

[2]This is abuse of notation, because we hope to present it as a morphism between schemes instead of a part of the functor σ^*.

$\coprod_{g\in G} X_L$ by

$$(\sigma,\tau)\mapsto\sigma$$
$$(\sigma,\tau)\mapsto\tau$$
$$(\sigma,\tau)\mapsto\sigma\tau$$

respectively. Then we can define the Galois descent for quasi-coherent sheaves.

Definition A.9. Let X be a scheme over K. Then a **Galois descent datum** (along $X_L \to X$) is a quasi-coherent sheaf $\mathcal{F}$ on X_L with a family of isomorphisms

$$\{\varphi_\sigma : T_\sigma^*\mathcal{F} \to \mathcal{F}\}_{\sigma\in G}$$

satisfying the cocycle condition

$$\varphi_\sigma \circ (T_\sigma^*\varphi_\tau) = \varphi_{\sigma\tau} \text{ for all } \sigma,\tau \in G.$$

Theorem A.10. Let X be a scheme over K. Then the category of quasi-coherent sheaves on X is isomorphic to the category of Galois descent data along $X_L \to X$.

This theorem is true due to the effectiveness of descent data along the fpqc covering Spec$L \to$ Spec$K$.

Any sheaf of algebra $\mathcal{A}$ over S can be realized as an relative affine scheme over S i.e. an affine morphism[3] $f : \mathbf{Spec}\mathcal{A} \to S$ such that for any open subset $U \subset S$, $\mathcal{A}(U) \cong \Gamma(f^{-1}(U); \mathbf{Spec}\mathcal{A})$. Therefore, descent data for relatively affine schemes can be encoded in descent data for quasi-coherent sheaves. Similarly, we also have relative projective construction **Proj** for sheaves of graded algebras over S, which means that descent data for relatively projective schemes can also be encoded in descent data for quasi-coherent sheaves.

Therefore, we can define descent data for schemes in an analogous way.

Definition A.11. A Galois descent datum along L/K is a L-scheme V with a family of isomorphism $\{\varphi_\sigma : \sigma^*V \to V\}_{\sigma\in G}$ satisfying the following cocycle condition: $\varphi_\sigma \circ (\sigma^*\varphi_\tau) = \varphi_{\sigma\tau}$ for all $\sigma,\tau \in G$ i.e. the following diagram commutes

$$\begin{tikzcd} (\sigma\tau)^*V \arrow[rr, "\varphi_{\sigma\tau}"] \arrow[dr, "\sigma^*\varphi_\tau"'] & & V \\ & \sigma^*V \arrow[ur, "\varphi_\sigma"'] & \end{tikzcd}$$

At least we know that if V is an affine or projective L-scheme, the Galois descent is effective. Moreover, if the descent datum on V is effective, then any clopen subsets of V with the associated restriction descent datum is also effective.

Theorem A.12. Let L/K be a finite Galois extension with Galois group G. Then any Galois descent datum $(V, \{\varphi_\sigma\}_{\sigma\in G})$ is effective if V is quasi-projective.

[3]A morphism $f : X \to Y$ between scheme is affine if for any affine open subscheme $U \subset Y$, $f^{-1}(U)$ is an affine open subscheme in X.

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