LOCALIZATION SEQUENCES OF HIGHER CHOW GROUPS OF A DVR

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Abstract. Levine gave an extension of Bloch's localization theorem for the higher Chow groups to schemes of finite type over a Dedekind domain.In particular, given a discrete valuation field (K, v) with the valuation ring $\mathcal{O}_K$ and the residue field k, Levine's localization sequence induces a boundary map $\mathrm{CH}^n(\mathrm{Spec}K, n) \xrightarrow{\partial} \mathrm{CH}^{n-1}(\mathrm{Spec}k, n-1)$. Using Nesterenko-Suslin's identification $\mathrm{CH}^n(\mathrm{Spec}F; n) \cong K_n^M(F)$ for any field F, we will show that this boundary map coincides with the residue boundary map ∂_v in the Milnor K-theories.

Contents

1. Recollection: the Milnor K-theory and higher Chow groups 1
1.1. Milnor's K-theory 1
1.2. Higher Chow groups 1
2. The computation of the boundary map 2
3. Applications 4
Acknowledgement 4
References 4

1. Recollection: the Milnor K-theory and higher Chow groups

1.1. Milnor's K-theory.

Definition 1.1. Let F be a field and its Milnor K-theory $K_*^M(F)$ is defined to be

$$\mathrm{Sym}_{\mathbb{Z}}(F^\times)/\{a \otimes 1 - a \mid a \in F \setminus 0, 1\}$$

where $\mathrm{Sym}_{\mathbb{Z}}$ means the free commutative algebra generated by the abelian group $F^\times$. We denote the image of $a_1 \otimes \cdots \otimes a_n$ by $\{a_1, \cdots, a_n\}$.

Lemma 1.2 ([Mil70])**.** Let $a_1, \cdots, a_n \in F^\times$. If $\sum_{i=1}^n a_i = 1$, then $\{a_1, \cdots, a_n\} = 0$.

Proposition 1.3 ([Mil70])**.** Let (K, v) be a valuation field with uniformizer t, valuation ring $\mathcal{O}_K$, and residue field k. Then there is a unique homomorphism

$$\partial_v : K_n^M(K) \to K_{n-1}^M(k)$$

such that for any $u_1, \cdots, u_n \in \mathcal{O}_K^\times$ and $x \in K^\times$, we have $\{x, u_2, \cdots, u_n\} \mapsto v(x)\{\bar{u}_2, \cdots, \bar{u}_n\}$, where $\bar{u}_i$ is the residue class of u_i in k.

1.2. **Higher Chow groups.** Now we recall the construction of higher Chow groups in the sense of Bloch [Blo86] and Levine [Lev01].

Let B be a regular Noetherian scheme of dimension 1. Let $X \xrightarrow{p} B$ be an irreducible B-scheme of finite type scheme. The dimension of X is defined in the following two cases. Let $\eta \in B$ be the image of the generic point along p.

(1) If η is a closed point, then $\dim X := \dim_{k(\eta)} X$.
(2) If η is not a closed point, then $\dim X := \dim_{k(\eta)} +1$.

Let X be an equidimensional scheme of dimension d and $V \subset X$ be an irreducible closed subscheme of X of dimensional i. The codimension $\mathrm{codim}_X V$ is defined to be $d - i$.

We say two closed subschemes V and W of X intersect properly, if for each irreducible component C of $W \cap V$

$$\dim C \leq \dim V + \dim W - \dim X$$

For $n \geq 1$, we define the n-algebraic simplex $\Delta_B^n := \mathrm{Spec}(\mathcal{O}_B[x_0, \cdots, x_n])/(\sum_i x_i - 1)$. The i-th (codimension 1) face of Δ_B^n is the closed subscheme cut by x_i. Given an equi-dimensional B-scheme of finite type, we let $z^n(X, i)$ be the free abelian group generated by all closed integral subschemes of codimension n in $X \times_B \Delta_B^i$, which intersect all the faces properly. This forms a simplicial abelian groups and we get the Bloch-Levine cycle complexes after doing Dold-Kan correspondence. The higher Chow groups of X are defined to be $\mathrm{CH}^q(X, p) := H_p(z^q(X, *))$.

Theorem 1.4 (Levine)**.** Suppose B is a DVR and X is a B-scheme of finite-type. Let $i \colon Z \to X$ be a closed embedding of codimension c and $j \colon U \to X$ be its open complement, then we have the following exact sequence

$$0 \longrightarrow z^{q-c}(Z, *) \xrightarrow{i_*} z^q(X, *) \xrightarrow{j^*} z^q(U, *)$$

and the cokernel of j^* is acyclic, which induces long exact sequences of higher Chow groups

$$\cdots \longrightarrow \mathrm{CH}^{q-c}(Z, p) \xrightarrow{i_*} \mathrm{CH}^q(X, p) \xrightarrow{j^*} \mathrm{CH}^q(X, p) \xrightarrow{\partial} \mathrm{CH}^{q-c}(Z, p-1) \longrightarrow \cdots$$

In this article, we focus on the case $X = \mathrm{Spec}\mathcal{O}_K$, $Z = \mathrm{Spec}k$ and $U = \mathrm{Spec}K$ for a discrete valuation field K. For dimension reasons, $\mathrm{CH}^p(\mathrm{Spec}K, q) = 0 = \mathrm{CH}^p(\mathrm{Spec}k, q)$ if $q > p$. The edge of non-trivial range of these higher Chow groups are $p = q = n \geq 0$.

Theorem 1.5 (Nesterenko-Suslin,Totaro)**.** For each $n \geq 0$ and a field F, there is a natural isomorphism $K_n^M(F) \cong \mathrm{CH}^n(\mathrm{Spec}F, n)$.

2. The computation of the boundary map

Let (K, v) be a valuation field with uniformizer t, valuation ring $\mathcal{O}_K$ and residue field k. The Milnor K-theory $K_n^M(K)$ is generated by the symbols of the form $\{t, u_2, \cdots, u_n\}$, where $u_i \in \mathcal{O}_K^\times$. According to Nesterenko-Suslin's construction [NS89], the symbol $u = \{t, u_2, \cdots, u_n\}$ corresponds to the zero cycle

$$\lambda_u = 1 - t - \sum_{i=2}^n u_i, \;\; W_u = (\frac{-t}{\lambda_u}, \frac{-u_2}{\lambda_u}, \cdots, \frac{-u_n}{\lambda_u}, \frac{1}{\lambda_u}) \subset \Delta_K^n.$$

More precisely, it is the prime ideal $I(W_u)$

$$(x_0 + \frac{t}{\lambda_u}, x_1 + \frac{u_2}{\lambda_u}, \cdots, x_{n-1} + \frac{u_n}{\lambda_u}, x_n - \frac{1}{\lambda_u})$$

in $K[x_0, \cdots, x_n]$ containing $(\sum_{i=0}^n x_i - 1)$. Note that $\Delta_{\mathcal{O}_K}^n \subset \Delta_K^n$ is an open subscheme. We let $\widetilde{W}_n$ be the closure of $W_u \cap \Delta_{\mathcal{O}_K}^n$ in $\Delta_{\mathcal{O}_K}^n$. In other words, $\widetilde{W}_u$ is the scheme-theoretic image of the locally closed embedding

$$W_u \hookrightarrow \Delta_K^n \hookrightarrow \Delta_{\mathcal{O}_K}^n$$

which means that the ideal $I(\widetilde{W}_u)$ is given by the kernel of the composition

$$\mathcal{O}_K[x_0, \cdots, x_n] \hookrightarrow K[x_0, \cdots, x_n] \to K[x_0, \cdots, x_n]/I(W_u).$$

Lemma 2.1. The cycle class $[\widetilde{W}_u]$ does not meet i-th face of $\Delta_{\mathcal{O}_K}^n$ when $i > 1$, and it meets the 0-th face if and only if λ_u is a unit. Moreover, the cycle class $[\widetilde{W}_u]$ is in $z^n(\mathrm{Spec}\mathcal{O}_K, n)$.

Proof. Let $\sigma_i\colon \Delta^{n-1}_{\mathcal{O}_K} \to \Delta^n_{\mathcal{O}_K}$ be the i-th face map $(x_0,\cdots,x_n) \mapsto (x_0,\cdots,x_{i-1},0,x_i,\cdots,x_n)$. Let f_i be the associated map on the rings of functions

$$\mathcal{O}_K[x_0,\cdots,x_n] \xrightarrow{f_i} \mathcal{O}_K[x_0,\cdots,x_{n-1}],\ x_j \mapsto \begin{cases} x_j, & j<i \\ 0, & j=i \\ x_{j-1}, & j>i \end{cases}$$

According to the definition, one has $(\lambda_u x_0 + t, \lambda_u x_1 + u_2, \cdots, \lambda_u x_{n-1} + u_n, \lambda_u x_n - 1) \subset I(\widetilde{W}_u)$. Therefore, for $i>0$, $\sigma_i^{-1}(\widetilde{W}_u)$ is empty, because $f_i(\lambda_u x_i + u_{i+1}) = u_{i+1}$ is a unit.

In the case of the 0-th face's intersection $\sigma_0^*[\tilde{W}_u]$, we divide it in two cases. If $v(\lambda_u)>0$, in other worlds, $\lambda_u = \varepsilon t^m$ for some $m \geq 1$ and $\varepsilon \in \mathcal{O}_K^\times$, then $t^{m-1}x_0 - \varepsilon^{-1} \in I(\widetilde{W}_u)$, and thus $f_0(I(\widetilde{W}_u)) = (\varepsilon^{-1}) = (1)$, which means that $[\widetilde{W}_u]$ does not meet the 0-th face.

If $v(\lambda_u)=0$ i.e. $\lambda_u \in \mathcal{O}_K^\times$, then we may lift the ideal $I(W_u)$ directly as an ideal in $\mathcal{O}_K[x_0,\cdots,x_n]$ and $I(\widetilde{W}_u) = (x_0 - \lambda_u^{-1}, x_1 + \lambda_u^{-1}u_2, \cdots, x_{n-1} + \lambda_u^{-1}u_n, x_n - \lambda_u^{-1}) \subset \mathcal{O}_K[x_0,\cdots,x_n]$. In this case, the ideal generated by $f(I(\widetilde{W}_u))$ is $(t, x_0 + \lambda_u^{-1}u_2, \cdots, x_{n-2} + \lambda_u^{-1}u_n, x_{n-1} - \lambda_u^{-1})$, whose quotient ring is k, and thus $\dim \sigma_0^{-1}(\widetilde{W}_u) = 0$. □

For now on, we abbreviate $z^*(\mathrm{Spec}R,*)$ by $z^*(R,*)$ and $\mathrm{CH}^*(\mathrm{Spec}R,*)$ by $\mathrm{CH}^*(R,*)$ for any commutative ring R.

Theorem 2.2. The boundary map $\mathrm{CH}^n(K,n) \to \mathrm{CH}^{n-1}(k,n-1)$ induced by the localization sequence $0 \to z^{n-1}(k,*) \xrightarrow{i_*} z^n(\mathcal{O}_K,*) \xrightarrow{j^*} z^n(K,*)$ coincides with the residue boundary map $K^M_n(K) \xrightarrow{\partial_v} K^M_{n-1}(k)$.

Proof. By unwinding the construction of the boundary map and following the notation in the beginning of this section, one consider the diagram,

$$\begin{tikzcd}
 & & \widetilde{W}_u \arrow[d, dash] & & & \\
0 \arrow[r] & z^{n-1}(k,n) \arrow[r] \arrow[d] & z^n(\mathcal{O}_K,n) \arrow[r] \arrow[d, "d_n"] & z^n(K,n) \arrow[r, dash] \arrow[d] & W_u \arrow[ull, dashed, mapsto, bend right] & \\
0 \arrow[r] & z^{n-1}(k,n-1) \arrow[r] \arrow[d, "\chi"'] & z^n(\mathcal{O}_K,n-1) \arrow[r] & 0 & (K^\times)^n \arrow[ul, "\theta"] \arrow[r, dash] \arrow[d, two heads] & \{t,u_2,\cdots,u_n\} \arrow[ul, mapsto] \\
 & K^M_{n-1}(k) & & & K^M_n(K) \arrow[lll, "\partial_v"'] &
\end{tikzcd}$$

Note that $[\tilde{W}_u]$ is an allowable cycle in good positions, according to Lemma 2.1. We divide the discussion of the boundary map in two cases.

Case 1: If $v(\lambda_u)=0$, then $d_n[\widetilde{W}_u] = \sigma_0^*[\widetilde{W}_u]$, cut by $(t, x_0+\lambda_u^{-1}u_2, \cdots, x_{n-2}+\lambda_u^{-1}u_n, x_{n-1}-\lambda_u^{-1})$, so it is on the special fiber over $\mathcal{O}_K$. Therefore, one can lift the cycle $\sigma_0^*[\widetilde{W}_u]$ to the zero cycle $(-\bar{u}_2/\bar{\lambda}_u, \cdots, \bar{u}_n/\bar{\lambda}_u, 1/\bar{\lambda}_u)$ in Δ_k^{n-1}. Using Nesterenko-Suslin's identification, this cycle corresponds to the symbol $\{\bar{u}_2,\cdots,\bar{u_n}\}$, which coincides with Proposition 1.3.

Case 2: If $v(\lambda_u)>0$, then $[\widetilde{W}_u]$ does not meet any face in $\Delta^n_{\mathcal{O}_K}$ and thus $\partial[W]=0$. Meanwhile, $v(\lambda_u)>0$ means that $\Sigma_{i=2}^n \bar{u}_i = 1$, and by Lemma [Mil70, Lemma 1.3], one has $\{\bar{u}_2,\cdots,\bar{u_n}\} = 0$ in this case, which also coincides with each other. □

Corollary 2.3. Following previous notation, one have

(1) $\mathrm{CH}^n(\mathcal{O}_K,n) = \ker(\partial_v\colon K^M_n(K) \to K^M_{n-1}(k))$,

(2) $\mathrm{CH}^n(\mathcal{O}_K, n-1) = \mathrm{coker}(\partial_v \colon K_n^M(K) \to K_{n-1}^M(k))$=0.

Proof. Now we adapt the closed embedding Spec$k \to$ Spec$\mathcal{O}_K$ to Levine's localization sequence:

$$0 \xrightarrow{i_*} \mathrm{CH}^n(\mathcal{O}_K, n) \xrightarrow{j^*} \mathrm{CH}^n(K, n) \xrightarrow{\partial} \mathrm{CH}^{n-1}(k, n-1)$$

Therefore, we can deduce that $\mathrm{CH}^n(\mathcal{O}_K, n) = \ker(\partial_v \colon K_n^M(K) \to K_{n-1}^M(k))$.

For another part of the long exact sequence, one has

$$\mathrm{CH}^n(K, n) \xrightarrow{\partial} \mathrm{CH}^{n-1}(k, n-1) \longrightarrow \mathrm{CH}^n(\mathcal{O}_K, n-1) \xrightarrow{j^*} 0$$

Thus one has $\mathrm{CH}^n(\mathcal{O}_K, n-1) = \mathrm{coker}(\partial_v \colon K_n^M(K) \to K_{n-1}^M(k))$. $\square$

3. Applications

We rewrite higher Chow groups in motivic cohomology groups by $\mathrm{H}^p(-;\mathbb{Z}(q)) := \mathrm{H}^q(-, 2q-p)$.

We assume K is a p-adic field with residue field k with characteristic p. According to the result by Gessier and Levine [GL00], one has

$$\mathrm{H}^i(k; \mathbb{F}_p(n)) = \begin{cases} K_n^M(k)/p, & i = n; \\ 0, & i \neq n. \end{cases}, \quad K_n^M(k)/p = \begin{cases} \mathbb{F}_p, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

Hence when $i > j$, $\mathrm{H}^{i+1}(\mathcal{O}_K; \mathbb{F}_p(j-1)) \cong \mathrm{H}^{i+1}(K; \mathbb{F}_p(j))$.

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