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THE STEENROD SQUARES ENCODE THE DATA OF HOMOTOPY COHERENT STRUCTURES

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Abstract. In algebraic topology, one often encounters diagrams of spaces that are commutative up to homotopy, rather than strictly commutative. However, by passing to the homotopy category, one loses the information of higher homotopies. This makes the corresponding algebraic invariants less effective to distinguish spaces. To give a more faithful algebraic picture for a geometric problem, it is desirable to devise machineries that capture higher homotopies. In this thesis, I show how the cup-i products and the Steenrod squares encode the data of higher homotopy types. From this perspective, I explain why the Steenrod squares and, more generally, cohomology operations for generalized cohomology theories work effectively as algebraic invariants for spaces in an attempt to understand the *raison d'être* of infinity-categorical algebra. This is based on investigating the literature and reorganizing theoretical and computational aspects of important tools in algebraic topology into an organic entirety through the theme of homotopy coherence. These include cohomology operations, simplicial sets, classifying spaces, and spectral sequences.

Contents

1. Introduction

To classify, simplify and compute with geometric objects, algebraic invariants are useful, especially homotopy invariants.

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Table 1. Methods of algebraic topology

Geometric objects		Algebraic objects
CW complexes		Numbers
Manifolds	Chosen invariants	Groups
Schemes	$\longrightarrow$	Rings
Data sets		Chain complexes
$\cdots$		$\cdots$
Geometric Morphisms		Algebraic Morphisms
Homotopy	$\longrightarrow$	Equality

Example 1.1 (The use of homology theory)**.** 2-dimensional sphere S^2 is not homotopy equivalent to 2-dimensional torus T^2:

$$H_1(S^2;\mathbb{Z}) = 0 \neq H_1(T^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

This means that there is no non-trivial one dimensional hole on S^2 while there are two non-trivial and unequivalent one dimensional hols on T^2.

However, homology theory may fail to distinguish spaces.

Example 1.2 (A blind spot of homology theory)**.** Let's compare $\mathbb{CP}^2$ and $S^2 \vee S^4$. The homology groups do not help.

Table 2. The homology groups of X and A with $\mathbb{Z}$ coefficient

	H_0	H_1	H_2	H_3	H_4	$\cdots$
$\mathbb{CP}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$S^2 \vee S^4$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Fortunately, we have a more powerful invariants to fix the blind spot.

Example 1.3 (The use of cohomology rings)**.** If we just take cohomology groups, there is no difference: However, by considering the cup product structure

Table 3. The cohomology groups of X and A with $\mathbb{Z}$ coefficient

	H^0	H^1	H^2	H^3	H^4	$\cdots$
$\mathbb{CP}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$S^2 \vee S^4$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

$$\smile : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$

$H^*(X)$ is a commutative graded ring, which gives sharper algebraic pictures than homology groups.

Back to the case that $\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}[u]/(u^3)$ and $S^2 \vee S^4$. We use the following facts to show that they are not homotopy equivalent.

(1) $H^*(\mathbb{CP}^2;\mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, where u is a generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$. In particular, $u^2 = u \smile u$ generates $H^4(\mathbb{CP}^2;\mathbb{Z})$.
(2) The cup product structure on $H^*(S^2 \vee S^4;\mathbb{Z})$ is trivial, namely, $u \smile v = 0$ for any two cohomology class u, v.

(3) $H^*(\mathbb{CP}^2;\mathbb{Z})$ is never isomorphic to $H^*(S^2 \vee S^4;\mathbb{Z})$.

Nevertheless, cohomology rings still may fail.

Proposition 1.1. *For any space X, the cup product structure on $H^*(\Sigma X)$ is trivial.*

Proof. The proof can be found in the step 5 of the proof of Theorem 7.2 $\square$

Example 1.4 (A blind spot of cohomology theory)**.** According to the previous problem, we cannot distinguish $\Sigma\mathbb{CP}^2$ and $\Sigma(S^2 \vee S^4) = S^3 \vee S^5$ by cohomology theory.

To cure the blindness of cohomology, we need to construct more invariants on cohomology rings. Thus we need **cohomology operations**.

Definition 1.5 (Cohomology operations)**.** Let n, m be two integers and let π, G be two abelian groups, a **cohomology operation** of type $(n, \pi; m, G)$ is a collection of functions $\varphi_X : H^n(X;\pi) \to H^m(X;G)$ for each CW-complex X such that for any continuous map $f : X \to X$, the following diagram commutes

$$\begin{tikzcd}
H^n(X;\pi) \arrow[r, "\varphi_X"] & H^q(X;G) \\
H^n(Y;\pi) \arrow[u, "f^*"] \arrow[r, "\varphi_Y"'] & H^m(Y;G) \arrow[u, "f^*"']
\end{tikzcd}$$

Clearly, the sum of two cohomology operations of the same type is still a cohomology operations. We denote the group of cohomology operations of type $(n, \pi; m, G)$ by $\mathcal{O}(n, \pi; m, G)$.

A **stable cohomology operation** of type (r, π, G) is a sequence of cohomology operations $\varphi_n \in \mathcal{S}tab(n, \pi; n + r; G)$ for $n = 1, 2, 3, \ldots$ such that for every X and every n, the following diagram commutes

$$\begin{tikzcd}
H^n(X;\pi) \arrow[r, "(\varphi_n)_X"] \arrow[d, "\Sigma"'] & H^{n+r}(X;G) \arrow[d, "\Sigma"] \\
H^{n+1}(\Sigma X;\pi) \arrow[r, "(\varphi_{n+1})_{\Sigma X}"'] & H^{n+r+1}(\Sigma X;G)
\end{tikzcd}$$

where Σ is the suspension isomorphism.

Let $\mathcal{S}tab(r; \pi, G)$ be the collection of stable cohomology operations of type $(r, \pi; G)$.

The Steenrod squares are significant stable cohomology operations that can help us fix the blind spot in Example 1.4.

Definition 1.6. The i-th Steenrod square Sq^i consists of stable cohomology operations

$$Sq^i : H^n(X;\mathbb{F}_2) \to H^{n+1}(X;\mathbb{F}_2)$$

for each $n \in \mathbb{N}$ satisfying the following axioms

(1) For any cocycle α, we have

$$Sq^i\alpha = \begin{cases} 0, & i > \dim\alpha, \\ \alpha^2, & i = \dim\alpha, \\ \alpha, & i = 0. \end{cases}$$

(2) The following Cartan's multiplication formula holds:

$$Sq^i(\alpha \smile \beta) = \sum_{p+q=i} Sq^p(\alpha) \smile Sq^q(\beta)$$

Theorem 7.2 shows the existence.

Example 1.7 (Cure the blindness of cohomology theory)**.** Let's back to the example of $\Sigma\mathbb{CP}^2$ and $S^3 \vee S^5$: Suppose there is a homotopy equivalence $f\colon S^3 \vee S^5 \to \Sigma\mathbb{CP}^2$, then we consider the Steenrod square

$$Sq^2\colon H^3(\Sigma\mathbb{CP}^2;\mathbb{Z}) \longrightarrow H^5(\Sigma\mathbb{CP}^2;\mathbb{Z})$$

Let u be a generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$, then by the suspension isomorphism, Σ^*u is a generator of $H^3(\Sigma\mathbb{CP}^2;\mathbb{Z})$. According to the definition, $Sq^2\Sigma^*u = \Sigma^*Sq^2u = \Sigma^*(u^2) \neq 0$, a generator of $H^5(\Sigma\mathbb{CP}^2;\mathbb{Z})$.

However, $f^*Sq^2\Sigma^*u = \Sigma^*Sq^2f^*u = \Sigma^*(f^*u)^2 = 0$, which leads to contradiction.

In this article, I will show how the Steenrod squares encode the data of homotopy coherence. Section 2 will show the significance of homotopy coherent structures (Theorem 2.4), and formulate it by using simplicial categories; Section 3 modify the category Ch of chain complexes to make it a simplicial category so that we can use the result mentioned in Section 2 on chain complexes; Section 4 will use the set-up in Section 3 to show that the homotopy coherent structure of the cup product will derive cup-i products on the cochain level; Section 5 will provide an intuitive way to describe cup-i products; Section 6 will show how to construct the Steenrod squares (on the cohomology level) by cup-i products; Section 7 will use spectral sequence to show the Steenrod squares exist and are uniquely determined by their properties.

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{Homotopy coherent structures}\\ \text{on the level of spaces}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{Cup-}i\text{ products}\\ \text{on the level of (co)chain complexes}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{Steenrod squares}\\ \text{on the level of cohomology theories}\end{array}}
\end{tikzcd}$$

2. Homotopy coherence and realization problems

Definition 2.1. Let A be a small category, a commutative diagram (of A-shape) is a functor $F\colon A \to \mathbb{S}\text{pace}$; a homotopy commutative diagram is a functor $G\colon A \to \text{Ho}(\mathbb{S}\text{pace})$.

Given a homotopy commutative diagram $F\colon A \to \text{Ho}(\mathbb{S}\text{pace})$, if there is a functor $G : A \to \mathbb{S}\text{pace}$ such that the composition $\pi \circ G : A \to \text{Ho}(\mathbb{S}\text{pace})$ is natural isomorphic to F, namely, there is a natural transformation $N: \pi \circ G \to F$, such that for any $f: x \to y$ in A, the following diagram commutes

$$\begin{tikzcd} G(x) \arrow[r, "N_x"] \arrow[d, "G(f)"'] & F(x) \arrow[d, "F(f)"] \\ G(y) \arrow[r, "N_y"'] & F(y) \end{tikzcd}$$

where N_x, N_y are homotopy equivalences. then we say G **is a realization of** F.

Problem 2.1 (The realization problem)**.** Given a homotopy commutative diagram, does the realization exists?

Example 2.2 (G-spaces and homotopy G-spaces)**.** Suppose G is a group, the associated groupoid $\mathcal{B}G$ is a category with one object $*$, and $\mathrm{Hom}(x, x) := G$ where the composition rule is given by the group multiplication.

A G-space is a functor $\mathcal{B}G \to \mathbb{S}\mathrm{pace}$. We may also say a space X is a G-space if there is a functor $\mathcal{B}G \to \mathbb{S}\mathrm{pace}$ such that X is the image of $*$. Similarly, a homotopy G-space is a functor $\mathcal{B}G \to \mathrm{Ho}(\mathbb{S}\mathrm{pace})$.

Let X be a G-space, if $f: Y \to X$ is a homotopy equivalence, then Y is a homotopy G-space. We may say X is a realization of Y.

In [Coo78], Cooke gave an answer to the realization problem of $\mathcal{B}G$-shaped diagram for some group G.

Theorem 2.3 (Cooke, 1978)**.** *A homotopy G-space can be realized by a G-space X if and only if the lifting problem 1 has a solution.*

$$\begin{tikzcd} & B\mathrm{Aut}(Y) \arrow[d, "B\tau"] \\ BG \arrow[ur, dashed] \arrow[r, "B\alpha"] & B\mathrm{Aut}_0(Y) \end{tikzcd} \tag{1}$$

where $\mathrm{Aut}(Y)$ *be the group of automorphisms of Y in* $\mathbb{S}\mathrm{pace}$*,* $\mathrm{Aut}_0(Y)$ *be the group of automorphisms of Y in* $\mathrm{Ho}(\mathbb{S}\mathrm{pace})$ *and* $\alpha: G \to \mathrm{Aut}_0(Y)$ *is determined by the homotopy group action.* $B: \mathrm{Abel} \to \mathbb{S}\mathrm{pace}$ *is the functor of classifying space.*

For the general case, the answer to the realization problem is given in [DKS89].

Theorem 2.4 (Dwyer-Kan-Smith,1989)**.** *A homotopy diagram has a realization of and only if it can be lifted to a* ***homotopy coherent*** *diagram.*

In brief, a diagram is homotopy coherent if it does not only have homotopies to make the diagram commute up to homotopy, but also have higher homotopies make the lower homotopies coherent. We use an example to interpret the homotopy coherent phenomenon.

Example 2.5. Let's consider a diagram

$$\omega := 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow \cdots$$

An ω-shaped diagram in $\mathbb{S}\mathrm{pace}$ consists of space X_k for $k \in \omega$ and morphisms $f_{i,k}: X_i \to X_k$ for $i < k$.

If it is a homotopy commutative diagram, then for any $i < j < k$, there is a homotopy $h_{i,j,k}: f_{i,k} \simeq f_{j,k} \circ f_{i,j}$.

This process specifies a path in $\mathrm{Map}(X_i, X_k)$ from vertex $f_{i,k}$ to $f_{j,k} \circ f_{i,j}$.

If it is homotopy coherent, then for any $i < j < k < l$, the chosen homotopies provides four paths in $\mathrm{Map}(X_i, X_l)$:

(2)
$$\begin{tikzcd}
f_{i,l} \arrow[r, dash, "h_{i,k,l}"] \arrow[d, dash, "h_{i,j,l}"'] & f_{k,l} \circ f_{i,k} \arrow[d, dash, "f_{k,l} \circ h_{i,j,k}"] \\
f_{j,l} \circ f_{i,j} \arrow[r, dash, "h_{j,k,l} \circ f_{i,j}"] & f_{k,l} \circ f_{j,k} \circ f_{i,j}
\end{tikzcd}$$

there is a 2**-homotopy to filling the square in** $\mathrm{Map}(X_i, X_l)$**.**

Similarly, for $i < j < k < l < m$, there are twelve paths and six 2-squares in $\mathrm{Map}(X_i, X_m)$ and then we can specify a 3-homotopy to filling in this cube.

(3)
$$\begin{tikzcd}
 & f_{i,m} \arrow[rrrr, "h_{i,j,m}"] \arrow[dl, "h_{i,j,m}"'] \arrow[dd, "h_{i,j,m}"] & & & & f_{j,m} \circ f_{i,j} \arrow[dd, "h_{j,k,m} \circ f_{i,j}"] \\
f_{l,m} \circ f_{i,l} \arrow[rrr, "f_{l,m} \circ h_{i,j,l}"'] \arrow[dd, "f_{l,m} \circ h_{i,k,l}"'] & & & f_{l,m} \circ f_{j,l} \circ f_{i,j} \arrow[urr, "h_{j,l,m} \circ f_{i,j}"] \arrow[dd, "f_{l,m} \circ h_{j,k,l} \circ f_{i,j}"'] & & \\
 & f_{k,m} \circ f_{i,k} \arrow[rrrr, "f_{k,m} \circ h_{i,j,k}"] \arrow[dl, "h_{k,l,m} \circ f_{i,k}"] & & & & f_{k,m} \circ f_{j,k} \circ f_{i,j} \arrow[dll, "h_{k,l,m} \circ f_{j,k} \circ f_{i,j}"] \\
f_{l,m} \circ f_{k,l} \circ f_{i,k} \arrow[rrr, "f_{l,m} \circ f_{k,l} \circ h_{i,j,k}"] & & & f_{l,m} \circ f_{k,l} \circ f_{j,k} \circ f_{i,j} & &
\end{tikzcd}$$

Proceeding the procedure, homotopy coherence means that all such n-homotopies exits! In other words, any such n-cubes in the mapping spaces can be filled by higher homotopies.

Note that there exists homotopy commutative diagram that is not homotopy coherent.

Example 2.6 (A homotopy commutative but not homotopy coherent diagram)**.** Let p be the Hopf fibration, i be inclusion of fiber at the based point and n is a degree map $e^{i\theta} \mapsto e^{in\theta}$:

(4)
$$\begin{tikzcd}
 & S^1 \arrow[dl, "n"'] \arrow[dr, "i"] \arrow[dd, "*" near start] & \\
S^1 \arrow[rr, "i"] \arrow[dr, "*"'] & & S^3 \arrow[dl, "p"] \\
 & S^2 &
\end{tikzcd}$$

Since $\pi_1(S^3)$ is trivial, let $\alpha : i \simeq i \circ n$ be the homotopy. However, $p \circ \alpha$ is not 2-homotopic to the constant homotopy $*$.

In summary, a homotopy commutative diagram just specifies some 2-dimensional simplicial complexes in the mapping spaces, where **vertices are objects, 1-simplexes are morphisms and 2-simplexes are homotopies exhibit the**

compositions, see the following diagram

(5)
$$\begin{tikzcd} 0 \arrow[rr, "f_{02}"] \arrow[dr, "f_{01}"'] & & 2 \\ & 1 \arrow[ur, "f_{12}"'] \arrow[u, phantom, "\Uparrow h_{012}"] & \end{tikzcd}$$

where $0, 1, 2$ are vertices, f_{01}, f_{12}, f_{02} are 1-simplexes and the homotopy h_{012} from $f_{12} \circ f_{01}$ to f_{02} is the 2-simplex.

At the same time, a homotopy coherent diagram specifies some ∞-dimensional acyclic simplicial complexes in the mapping spaces, where higher simplexes exhibit higher homotopies, for example, recall the diagram 2, we may write it into

(6)
$$\begin{tikzcd} & l & \\ & & j \arrow[ul] \arrow[d] \\ i \arrow[uur] \arrow[urr] \arrow[rr] & & k \arrow[uul] \end{tikzcd}$$

where the morphisms $f_{i,j}, f_{i,k}, f_{i,l}, f_{j,k}, f_{j,l}, f_{k,l}$ corresponds to 1-simplexes $[i, j], \ldots, [k, l]$, the homotopies $h_{i,j,k}, \ldots, h_{j,k,l}$ correspond to 2-simplexes $[i, j, k], \ldots, [j, k, l]$ and the 2-homotopy corresponds to the 3-simplex $[i, j, k, l]$.

By observation, given a homotopy diagram $F : A \to \mathrm{Ho}(\mathbb{S}\mathrm{pace})$, for each 2-simplex in the nerve of A (see Section 7), there is a 2-simplex to fill the triangle just like the diagram 5. For more geometric intuition, we consider

Definition 2.7 (The classifying space of a small category)**.** Let $\mathcal{C}$ be a small category, the classifying space $B\mathcal{C}$ of $\mathcal{C}$ is defined by

$$B\mathcal{C} := |N_\bullet \mathcal{C}| = \coprod_{n \geq 0} \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C}) \times |\Delta_n| / \sim$$

and this is a functor $B\colon \mathrm{Cat} \to \mathbb{S}\mathrm{pace}$.

Remark 2.8. Let G be an Abelian group, then the classifying space BG of the G is equivalent to the classifying space $B(\mathcal{B}G)$ of the category $\mathcal{B}G$.

We expect the realization problem can be converted to such a lifting problem

(7)
$$\begin{tikzcd} & B\mathbb{S}\mathrm{pace} \arrow[d, "B\pi"] \\ BA \arrow[ur, dashed] \arrow[r, "BF"] & B\mathrm{Ho}(\mathbb{S}\mathrm{pace}) \end{tikzcd}$$

and the diagram 1 is just a special case. **However, we cannot do this because** $\mathbb{S}\mathrm{pace}$ **and** $\mathrm{Ho}(\mathbb{S}\mathrm{pace})$ **are not small categories. To provide some insight, we may assume the nerve of a large category makes sense.** Then if the diagram $F : A \to \mathrm{Ho}(\mathbb{S}\mathrm{pace})$ has a realization, then the lifting problem 7 has a solution. Conversely, if the lifting problem 7 has a solution, there is no hard to pass to the following diagram

$$
\begin{tikzcd}
 & \mathrm{Sing}_\bullet(|N_\bullet\mathbb{S}\mathrm{pace}|) \arrow[d, "\mathrm{Sing}_\bullet(B\pi)"] \\
\mathrm{Sing}_\bullet(|N_\bullet A|) \arrow[ur, dashed] \arrow[r, "\mathrm{Sing}_\bullet(BF)"] & \mathrm{Sing}_\bullet(|N_\bullet \mathrm{Ho}(\mathbb{S}\mathrm{pace})|)
\end{tikzcd}
\tag{8}
$$

Since $(|-|, \mathrm{Sing})$ are Quillen equivalences, there is a homotopy commutative diagram in sSet:

$$
\begin{tikzcd}
 & N_\bullet\mathbb{S}\mathrm{pace} \arrow[d, "N_\bullet\pi"] \\
N_\bullet A \arrow[ur] \arrow[r, "N_\bullet F"] & N_\bullet \mathrm{Ho}(\mathbb{S}\mathrm{pace})
\end{tikzcd}
\tag{9}
$$

Since the nerve functor $N_\bullet$ is fully faithful, the diagram 9 is actually a realization of F.

Remark 2.9. A homotopy commutative diagram can specify a 2-dimensional subcomplex of $|N_\bullet\mathbb{S}\mathrm{pace}|$ i.e. each 2-simplex in N_2A specify a homotopy in $\mathbb{S}\mathrm{pace}$ to witness a composition, see the diagram 5. Hence $F: A \to \mathrm{Ho}(\mathbb{S}\mathrm{pace})$ can specify a map $BA \to \mathrm{Sk}_2 B\mathbb{S}\mathrm{pace}$ and the diagram 7 is actually an extension problem. The diagram is homotopy coherent if and only if there is no obstruction. A brief introduction to the obstruction theory can be found in Section 7.

Since the nerve of a large category may not make sense, to formulate homotopy coherent phenomenon more precisely, we need the following definitions to show how higher homotopies be coherent.

Definition 2.10 (Simplicial category)**.** A simplicial category $\mathcal{C}_\bullet$ is category enriched by simplicial sets. The category of simplicial categories is denoted by sCat.

Definition 2.11 (Homotopy in a simplicial category)**.** Given a simplicial category $\mathcal{C}_\bullet$, morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}_\bullet}(X,Y)_0$ are homotopic if there is an 1-simplices in $\mathrm{Hom}_{\mathcal{C}_\bullet}(X,Y)_1$ whose boundary is f and g. Hence we can define the homotopy category $\mathrm{Ho}(\mathcal{C}_\bullet)$ by quotient the homotopy relation.

Example 2.12. Note that $\mathbb{S}\mathrm{pace}$ is enriched by Top (see Section 7). Since there is a lax monoidal functor Sing, $\mathbb{S}\mathrm{pace}$ is a simplicial enriched category by base changing (see Remark 7.20).

Next we will show for any small category, there will be a simplicial resolution for this category, namely, a suitable simplicial category substitution.

Construction 2.13 (Simplicial resolution)**.** First to define a cosimplicial object in sCat. We define the simplicial category $C([n])$ associated to $[n]$ by

$$
\mathrm{Hom}^{\mathrm{sSet}}_{[n]}(i,j) = \begin{cases} N_\bullet P_{i,j}, & i \le j, \\ \emptyset, & i > j. \end{cases}
$$

where $P_{i,j} := \{I \subset \{i, i+1, \dots, j-1, j\} \mid i, j \in I\}$ is a poset ordered by inclusion (as a category). Thus by Proposition 7.2, there is an adjunction

$$
(C, N_{\Delta\bullet}) \colon \mathrm{sSet} \rightleftharpoons \mathrm{sCat}
$$

where $N_{\Delta_\bullet}$ is called the **homotopy coherent nerve** functor and C is called **simplicial thickening**. Given a small category $\mathcal{C}$, the simplicial resolution is defined by

$$\mathcal{C}_\bullet := C(N_\bullet \mathcal{C})$$

More specifically, let U be the functor from Cat to the category of reflexive direct graphs defined by forgetting the composition law and given a reflexive directed graph G, let $F(G)$ be the category freely generated by G. By composition, the simplicial resolution is

$$\mathcal{C}_\bullet : \qquad FU\mathcal{C} \;\substack{\twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow}\; FUFU\mathcal{C} \;\substack{\twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow}\; (FU)^3\mathcal{C} \;\substack{\twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow \\ \longrightarrow \\ \twoheadleftarrow}\; (FU)^4\mathcal{C} \qquad \cdots$$

By the composition law, there is an **augmentation** $\epsilon\colon \mathcal{C}_\bullet \to \mathcal{C}$. More details are [Rie18].

Example 2.14. Given a group G, the universal principal G-bundle EG can be given by $EG \simeq |\mathcal{B}_\bullet G|$, see [May99], Chapter 16.

Definition 2.15 (Homotopy coherent diagram)**.** Suppose A is a small category, a homotopy coherent diagram is a simplicial functor $A_\bullet \to \mathbb{S}\mathrm{pace}$.

We say a homotopy commutative diagram $F : A \to \mathrm{Ho}(\mathbb{S}\mathrm{pace})$ is homotopy coherent if there is a lifting

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d, "\epsilon"'] & \mathbb{S}\mathrm{pace} \arrow[d, "\pi"] \\ A \arrow[r, "F"] & \mathrm{Ho}(\mathbb{S}\mathrm{pace}) \end{tikzcd} \tag{10}$$

Remark 2.16. For any simplicial category $\mathcal{S}$, we can define homotopy commutative diagrams and homotopy coherent diagrams in an analogous way.

3. The category of chain complexes is a simplicial category

A geometric complex K means a simplicial complex or a CW complex. By taking simplicial chain complex or cellular complex, we may identify a geometric complex with a chain complex $C_\bullet(K)$. Sometimes I may abbreviate $C_\bullet(K)$ by K and there is no harm.

The group $C_q(K)$ of **q-chains** is the free abelian group generated by the q-cells and the **boundary operator** is denoted by

$$\partial\colon C_q(K) \to C_{q-1}(K)$$

and $\partial \circ \partial = 0$. Suppose σ is a cell in K and τ is its face, then we may write $\tau < \sigma$. **The Kronecker index** $\mathrm{In}(c)$ of a 0-chain $c = \sum a_i x_i$ is defined to be $\sum a_i$, where x_i are 0-cells. We denote $Z_q(K)$, $B_q(K)$ and $H_q(K)$ the group of **q-cycles**, the group of **q-boundaries** and **the group of q homology classes**, respectively. We say two elements in $C_q(K)$ are **homologic** if they are different from a boundary.

Definition 3.1 (Acyclic complex)**.** A complex K is acyclic if $H_q(K) = 0$ for $q > 0$.

For any abelian group G, the cohomology with coefficient G is defined to be

$$C^q(K; G) := \mathrm{Hom}(C_q(K), G)$$

and in this way we have a cochain complex $C^\bullet(K;G)$ with coefficient G. The value of $u \in C^q(K;G)$ on $c \in C_q(K)$ is denoted by $u \cdot c$. The dual of boundary operators are **coboundary operators** denoted by $\delta\colon C^q(K;G) \to C^{q+1}(K;G)$.

If L is a subcomplex of K, then $C^q(K,L;G)$ is the subgroup of q-chains that are zero on cells of L. Similarly, we can define $Z^q(K,L;G)$, $B^q(K,L;G)$ and $H^q(K,L;G)$ respectively.

A chain map ϕ is a sequence of homomorphisms

$$\phi_q\colon C_q(K) \to C_q(K')$$

such that $\phi_q\partial = \partial\phi_{q+1}$ for any q and $\mathrm{In}(\phi c) = \phi\mathrm{In}(c)$. The category of chain complexes of $\mathbb{Z}$-modules is denoted by Ch. If we replace $\mathbb{Z}$ by a commutative ring, the notation is Ch_R.

Suppose f is a continuous map from K to K', due to the existence of simplicial approximation and cellular approximation, there exists a simplicial or cellular map f' such that $f' \simeq f$ and the induced may f_* can be given by f' directly.

Let C, D be two chain complexes and $f, g\colon C \to D$ be two chain maps.

Definition 3.2 (Chain homotopy I)**.** We say f, g are chain homotopic if there exits a collection of homomorphisms

$$\{h_i\colon C_n \to D_{n+1}\}_{n=0}^{\infty}$$

such that

$$\partial \circ h_i + h_{i-1} \circ \partial = g_i - f_i$$

Definition 3.3. We define a tensor product between C and D by

$$(C \otimes D)_n := \sum_{i+j=n} C_i \otimes D_j$$

and

$$\partial(c_i \otimes d_j) := \partial c_i \otimes d_j + (-1)^i c_i \otimes \partial d_j, \text{ for } c_i \in C_i \text{ and } d_j \in D_j.$$

The diagram is

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & C_n \arrow[r, "\partial"] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, dashed, "h"'] & C_{n-1} \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[dl, dashed, "h"'] \\
D_{n+1} \arrow[r, "\partial"] & D_n \arrow[r, "\partial"] & D_{n-1}
\end{tikzcd}$$

Definition 3.4 (Interval chain complex)**.** The interval chain complex $I_\bullet$ is defined by

$$\begin{tikzcd}
 & I_0 \arrow[d, Rightarrow] & I_1 \arrow[l] \arrow[d, Rightarrow] & I_2 \arrow[l] \arrow[d, Rightarrow] & \cdots \arrow[l] \\
0 & \mathbb{Z}a \oplus \mathbb{Z}b \arrow[l] & \mathbb{Z}u \arrow[l, "\partial_0"'] & 0 \arrow[l, "\partial_1"'] & \cdots \arrow[l]
\end{tikzcd}$$

where $\partial\colon u \mapsto a - b$.

Sometimes, for convenience, we may just write I by regarding it as a simplicial complex of the topological interval I.

Definition 3.5 (Chain homotopy II)**.** We say f, g are chain homotopic if there is a chain map $h'\colon C \otimes I_\bullet \to D$ such that $h'|_{C\otimes a} = f$ and $h'|_{C\otimes b} = g$.

The second definition of chain homotopy is more similar to the definition of topological homotopy and actually these two definitions of chain homotopy coincides by setting

$$h_i(c_i) = h'_i(c_i \otimes u), \text{ for } c_i \in C_i$$

Similarly, we can define chain homotopy on cochain complexes dually.

Definition 3.6. An operation of degree i from K to K' is defined to be a sequence of homomorphisms

$$D_i : C_q(K) \to C_{q+i}(K')$$

for all q and commutes with boundary maps. Let O_i be the set of all operations of degree i and it forms an additive group naturally. We define the boundary operator $\omega : O_i \to O_{i-1}$ by

$$(\omega D_i)c = \partial D_i c + (-1)^{i+1} D_i \partial c \tag{11}$$

Clearly, $\omega\omega = 0$ and the **operator complex** is defined by $(\{O_i\}, \omega)$. Specifically, the operator complex from K to K' is denoted by $O(K, K')$.

Proposition 3.1. *If D_i is an i-cycle in the operator complex, then D_i carries cycle into cycles, boundaries into boundaries, and thereby induces homomorphisms $H_q(K) \to H_{q+i}(K')$. If D_i, D'_i are homologous cycles, then for any i chain $c \in K_q$, $D_i c$ and $D'_i c$ are homologous.*

Sketch proof. Use the equation 11, the proof is straightforward. □

Definition 3.7. A 0-cycle D_0 in the operator complex has an **index** if there is an integer k such that $\mathrm{In}(D_0 c) = k\mathrm{In}(c)$ for any $c \in C_0(K)$ and k is the index. In particular, D_0 has index 1 if and only if D_0 is a chain map.

Proposition 3.2. *Let W be a complex, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Ch}}(W, O(K, K')) \cong \mathrm{Hom}_{\mathrm{Ch}}(W \otimes C_\bullet(K), C_\bullet(K'))$$

Sketch proof. The isomorphism is given by

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{Ch}}(W, O(K, K')) & \longrightarrow & \mathrm{Hom}_{\mathrm{Ch}}(W \otimes C_\bullet(K), C_\bullet(K')) \\ f & \longmapsto & [w_q \otimes c_q \mapsto f(w_q) \cdot c_q] \end{array}$$

The pattern is similar to

$$\mathrm{Hom}_{\mathrm{Mod}}(M \otimes N, P) \cong \mathrm{Hom}_{\mathrm{Mod}}(M, \mathrm{Hom}_{\mathrm{Mod}}(N, P))$$

in the category of modules. □

Now we have shown that the category Ch is enriched by itself by considering the operator complexes (see Definition 3.6). Then by base changing, Ch is a simplicial category via Dold-Kan correspondence (see Example 7.14 and Section 7.21). In this way, we can discuss homotopy coherent diagrams and the realization problems in Ch.

4. From cup products to cup-i products

Suppose K and K' are two cell complexes, the product space $K \times K'$ has a natural cell structure given by $\sigma \times \sigma'$, where σ is a cell in K and $K\sigma'$ is a cell in K'. The product of relative complexes is defined by

$$(K, L) \times (K', L') := (K \times K', K \times L' \cup L \times K')$$

Theorem 4.1 (Eilenberg-Zilber)**.** *$C_\bullet(K \times K') \simeq C_\bullet(K) \otimes C_\bullet(K')$ naturally.*

Definition 4.2 (Cross product)**.** Let G_1, G_2, G_3 be three abelian groups and a bilinear map $G_1 \times G_2 \to G_3$, the **cross product** is a collection of bilinear pairing defined by

$$\begin{array}{rrcl} \times: & C^p(K, L; G_1 \times C^q(K', L'; G_2) & \longrightarrow & C^{p+q}(K \times K', K \times L' \cup L \times K'; G_3) \\ & (u \times u') \cdot (\sigma \times \sigma') & \longmapsto & u(\sigma)u'(\sigma') \end{array}$$

Note that the cross products between cocycles are still cocycles. Hence cross products can be lifted to cohomology.

Definition 4.3 (Cup product)**.** Let K be a cell complex and we define a diagonal map

$$\begin{array}{rrcl} D: & K & \longrightarrow & K \times K \\ & x & \longmapsto & (x, x) \end{array}$$

Then the diagonal map induces $D_*\colon H_*(X) \to H_*(X \times X)$ and $D^*\colon H^*(X \times X) \to H^*(X)$. The cup product $\smile$ on $H^*(X)$ is defined by

$$\begin{tikzcd} H^p(X) \times H^q(X) \arrow[r, "\times"] \arrow[dr, "\smile"'] & H^{p+q}(X \times X) \arrow[d, "D^*"] \\ & H^{p+q}(X) \end{tikzcd}$$

The cross product is clear, hence to compute cup products, we need to compute Δ^* or Δ_*. Recall that if f is a cellular or simplicial map, then f_* is explicit. However, *for any nontrivial cell complex K, the diagonal map D is never cellular or simplicial.* Thus we need to find a good simplicial or cellular approximation of Δ and **Alexander-Whitney approximation** is what we need.

We begin with standard simplex. Let Δ^n be a standard n-simplex with ordered vertices $v_1, \ldots, v_n$. We now try to give a suitable cell structure for Δ^n. We rename v_i by v_{ii} and let $v_{ij} = \frac{v_{ii}}{2} + \frac{v_{jj}}{2}$, i.e. the middle point of the edge $[v_{ii}, v_{jj}]$ for $0 \leq i < j \leq n$. The set of 0-cells is $\{v_{ij}\}$ for the new simplicial structure. Then define a $(p+1)(q+1)$-cell be a prism spanned by

$$\{a_{ij} \mid i = i_0 < \cdots < i_p,\ j = j_0 < \cdots < j_q,\ i_p \leq j_0\}$$

These prisms give a new cell structure of Δ^n. To distinguish them, we denote the new one by $P(\Delta^n)$.(*Warning: this cell structure relies on the order of vertices.*)

Definition 4.4. The p**-front face** of Δ^n, denoted by ${}_p\Delta^n$ is the closed simplex(cell) spanned by $v_0, \ldots, v_p$; the $(n-p)$**-back face** of Δ^n_{n-p} is the closed simplex(cell) spanned by $v_p, \ldots, v_n$.

Proposition 4.1. *There is a cellular map $D_0\colon \Delta^n \to \Delta^n \times \Delta^n$ by mapping Δ^n homeomorphically onto $\cup_{p=0}^n ({}_p\Delta^n \times \Delta^n_{n-p})$ and $P(\Delta^n)$ has the same cell structure* *as* $\cup_{p=0}^{n}({}_p\Delta^n \times \Delta^n_{n-p}) \subset \Delta^n \times \Delta^n$ *via* Δ' *as subcomplex. On the chain level, the morphism is*

$$\begin{array}{rrcl} D_0': & C_\bullet(\Delta^n) & \longrightarrow & C_\bullet(\Delta^n \times \Delta^n) \\ & \Delta^n & \longmapsto & \sum_{p=0}^{n}({}_p\Delta^n \times \Delta^n_{n-p}) \end{array}$$

Proof. Let's consider the barycentric coordinate of Δ^n. Note that for any point $x \in \Delta^n$, it can be written as the form $\sum_{i=0}^{n} x_i v_i$, $\forall x_i \geq 0$ and $\sum_i x_i = 1$. Now we claim that x can be written in a form

$$\sum_{i=0}^{n} x_i v_i = \frac{1}{2}\sum_{i=0}^{p} y_i v_i + \frac{1}{2}\sum_{i=p}^{n} z_i v_i$$

where $y_i, z_i > 0$ and $\sum y_i = \sum z_i = 1$ for some p. The principle to find such p is that there exists a unique $0 \leq p \leq n$ such that

$$\sum_{i=0}^{p-1} x_i \leq \frac{1}{2} \text{ and } \sum_{i=0}^{p} x_i \geq \frac{1}{2}$$

Therefore, for any $x \in \Delta^n$, x can be written in form of $\frac{y}{2} + \frac{z}{2}$ form some $p \in {}_p\Delta^n$ and $z \in \Delta^n_{n-p}$ for the unique p.

Then we define

$$\begin{array}{rrcl} D_0: & \Delta^n & \longrightarrow & \Delta^n \times \Delta^n \\ & x & \longmapsto & (y, z) \end{array}$$

Clearly, D_0 is a cellular embedding with $D_0(v_{ij}) = (v_{ii}, v_{jj})$. Hence

$$\cup_{p=0}^{n}({}_p\Delta^n \times \Delta^n_{n-p}) \cong P(\Delta^n)$$

via this embedding. Specifically, the map is

$$D_0 \colon [v_0, \cdots, v_n] \mapsto \sum_{p=0}^{n} [v_0, \cdots, v_p] \times [v_p, \cdots, v_n] \tag{12}$$

□

Proposition 4.2. *The simplicial (cellular) map D_0 is homotopic to the diagonal map D.*

Proof. The homotopy is just given by a linear homotopy. □

For general case, suppose K is an ordered simplicial complex, define $D_0 \colon K \to K \times K$ by mapping each ordered simplex of K in the previous way. If τ is a common face of simplexes L and L', then $D_0|_L = D_0|_{L'}$ clearly by the definition. This map is called **Alexander-Whitney map** and the cup product is defined to by

$$\phi \smile \psi \cdot \sigma := (\phi \times \psi) \cdot D_0(\sigma)$$

Motivation 4.5. Let $\mathbb{F}_2$ acts on $X \times X$ by permuting the coordinates and let $T \colon (x, y) \mapsto (y, x)$ be the generator of this action. Note that $T \circ D = D$, but $T \circ D_0 \neq D_0$, though they are homotopic. Actually, any choice of simplicial approximation D is not invariant under the composition with T. There is a lack of symmetry when doing approximation, which means that we lose some information if we just identify

D_0 and D. More specifically, the following diagram is homotopy-commutative and is not strictly commutative

$$\begin{tikzcd} & X \times X \arrow[dd, "T"] \\ X \arrow[ur, "D_0"] \arrow[dr, "D_0"'] & \\ & X \times X \end{tikzcd} \tag{13}$$

which means that D_0 is not $\mathbb{F}_2$-equivariant but homotopically $\mathbb{F}_2$-equivariant if we let $\mathbb{F}_2$ acts on X trivially, and the cohomology rings lose the information of this symmetry by modulo homotopy. Thus our next goal is to measure the deviation from the symmetry.

Actually, it is essentially a realization problem in Ch. By taking chain complex of the diagram 13, there is a homotopy commutative diagram in Ch

$$\begin{tikzcd} & C_\bullet(X) \otimes C_\bullet(X) \arrow[dd, "T"] \\ C_\bullet(X) \arrow[ur, "D_0"] \arrow[dr, "D_0"'] & \\ & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd} \tag{14}$$

and this diagram is not strict commutative, which is the lack of symmetry. If there is a realization of the diagram 14, then the difference between the homotopy coherent diagram and the original diagram is the deviation from the symmetry. In the next part, we will show that the diagram 14 is homotopy coherent. The main reference of the following argument are [Ste52] and [Ste72].

To show it is homotopy coherent, namely, the existence of some higher homotopies, we need the following set-up.

Definition 4.6 (Carrier)**.** A **carrier** from complexes pair (K, L) to (K', L') is a function which assigns to each cell σ of K a non-trivial subcomplex $C(\sigma)$ of K such that $\sigma \in L$ implies $C(\sigma) \subset L'$ and if $\tau < \sigma$, then $C(\tau) \subset C(\sigma)$. A carrier is acyclic, if $C(\sigma)$ is acyclic for each cell $\sigma \in K$.

We say **a carrier carries a chain homotopy** h if for each cell σ, $h(\sigma) \in C(\sigma)$. Similarly, **a carrier carries a chain map** ϕ if $\phi(\sigma) \in C(\sigma)$.

Lemma 4.7 (Acyclic carrier lemma)**.** *If C is an acyclic carrier $K \to K'$, then C carries a chain map ϕ; and, if ϕ, ψ are two chain maps carried by C, then ϕ is homotopic to ψ.*

Proof. We construct such ϕ inductively on the dimension. First, for each 0-cell $\sigma \in K$, we just let $\phi(\sigma) \in C(\sigma)$ with index 1, for example, a 0-cell in K'. Then we can extend it to a homomorphism from $C_0(K)$ to $C_0(K')$. Suppose we have already define $\phi : C_n(K) \to C_n(K')$ for $n < q$, we need to construct a homomorphism $C_{n+1}(K) \to C_{n+1}(K')$. Let σ be a q-cell, then $\partial\sigma = \sum a_i c_i$, where c_i is a face of q-cell. Since $\sum a_i c_i$ is a cycle, $\partial\sigma$ is also a cycle by the inductive hypothesis and $\sum a_i \phi(c_i) \in C(\sigma)$. Then there is a chain $\phi(\sigma) \in C(\sigma)$ such that $\partial\phi(\sigma) = \phi(\partial\sigma)$, because $C(\sigma)$ is acyclic, namely each cycle is a boundary.

Next, we prove any two chain map ϕ, ψ carried by C are homotopic. We first write down the diagram

$$\begin{tikzcd} C_{q+1}(K) \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_q(K) \arrow[r, "\partial"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] & C_{q-1}(K) \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\psi"] \\ C_{q+1}(K') \arrow[r, "\partial"] & C_q(K') \arrow[r, "\partial"] & C_{q-1}(K') \end{tikzcd}$$

We construct the chain homotopy inductively on the dimension of cells. Since $\mathrm{In}(\phi\sigma) - \mathrm{In}(\psi\sigma) = 0$, we can find an 1-chain $h(\sigma) \in C(\sigma)$ such that $\partial h(\sigma) = \phi\sigma - \psi\sigma$, due to the acyclicness. Now we suppose for each n-cell τ, $n < q$, we have such $h(\tau)$ to exhibit the chain homotopy at lower dimension, then we need to find $h(\sigma)$ such that $\partial h(\sigma) + h(\partial\sigma) = \phi\sigma - \psi\sigma$. Note that $\phi\sigma - \psi\sigma - h(\partial\sigma)$ is a cycle, because $\partial(\phi\sigma - \psi\sigma) = \phi(\partial\sigma) - \psi(\partial\sigma)$ and by inductive hypothesis

$$\phi\sigma - \psi\sigma - h(\partial\sigma) = \phi(\partial\sigma) - \psi(\partial\sigma) - h(\partial\sigma) = \partial h(\partial\partial\sigma) = 0$$

Since $C(\sigma)$ is acyclic, we can find $h(\sigma) \in C(\sigma)$ such that

$$\partial h(\sigma) = \phi\sigma - \psi\sigma - h(\partial\sigma)$$

which is what we need. □

Definition 4.8. Let C be a carrier from K to K', the operator complex $O(C)$ associated to C is defined by

$$O(C)_q := \{D_q \in O_q \mid D_q(\sigma) \in C(\sigma), \forall \sigma \in C_q(K)\}$$

Lemma 4.9. *Let C be an acyclic carrier from K to K', then the associated operator complex $O(C)$ contains 0-cycle of index 1, and $O(C)$ is acyclic.*

Sketch proof. The proof of this lemma is similar to the proof of Theorem 4.7. □

Now we use this set-up to show the diagram 14 is homotopy coherent and see how it realize.

Let σ be an n-cell in X, let $\bar{\sigma}$ be the subcomplex containing all the faces of σ and it is acyclic. Let $C(\sigma) = \bar{\sigma} \otimes \bar{\sigma}$. By the definition, this forms a carrier from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$. Moreover, C is an acyclic carrier and T-invariant, namely, $TC(\sigma) \subseteq C(\sigma)$.

Since both D_0 and TD_0 are carried by C, by Lemma 4.7, they are homotopic. We let D_1 be a chain homotopy from D_0 to TD_0 carried by C. More specifically, for any n-cell σ in X, $D_1(\sigma)$ is in $C(\sigma)$ such that

$$\partial D_1(\sigma) + D_1(\partial\sigma) = TD_0(\sigma) - D_0(\sigma)$$

or

$$\partial D_1(\sigma) = TD_0(\sigma) - D_0(\sigma) - D_1(\partial\sigma) \tag{15}$$

Similarly,

$$\partial TD_1(\sigma) = D_0(\sigma) - TD_0(\sigma) - TD_1(\partial\sigma) \tag{16}$$

Notice that $D_1 + TD_1$ is a homotopy of D_0 around a circuit back to itself (the addition between chain homotopies in Ch means the join of homotopies) and $D_1(\sigma) + TD_1(\sigma) \in C(\sigma)$ for each cell σ. Since both D_1 and the constant homotopy of D_0 are carried by the acyclic carrier C, apply Lemma 4.7 again, and there is a chain homotopy D_2 from $D_1 + TD_1$ to the constant homotopy of D_0 carried by C. Specifically, for any n-cell σ, there is an $n+2$-cell $D_2(\sigma)$ such that

$$\partial D_2(\sigma) = D_1(\sigma) + TD_1(\sigma) + D_2(\partial\sigma)$$

Now observe that $D_2 - TD_2$ is a homotopy from $D_1 + TD_1$ to itself. Similarly, $D_2 - TD_2$ is homotopic to the constant homotopy of $D_1 + TD_1$, namely, there exists D_3 such that

$$\partial D_3(\sigma) = D_2(\sigma) - TD_2(\sigma) - D_3(\partial\sigma)$$

Repeat the procedure inductively, then we have $\{D_n\}_{n=0}^{\infty}$ to exhibit higher homotopies. Note that D_n is an operation of degree n from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$. Recall Definition 3.6 and the operator boundary 11,

$$\omega D_i = D_{i-1} + (-1)^{i+1} TD_{i-1}$$

We let W be the subcomplex of $O(C)$ from $C_\bullet(X)$ to $C_\bullet(X) \otimes C_\bullet(X)$ and W_n is freely generated by D_i and TD_i (since C is T-invariant, TD_i is also in $O(C)$).

Then according to Proposition 3.2, the inclusion map $W \hookrightarrow O(C)$ uniquely determined a chain map

$$\begin{array}{rrcl} \phi: & W \otimes C_\bullet(X) & \longrightarrow & C_\bullet(X) \otimes C_\bullet(X) \\ & D_i \otimes \sigma & \longmapsto & D_i(\sigma) \end{array} \tag{17}$$

which is realization of the diagram 14, because the diagram

$$\begin{tikzcd} W \otimes C_\bullet(X) \arrow[r, "\phi"] \arrow[d, "T\otimes\mathrm{id}"'] & C_\bullet(X) \otimes C_\bullet(X) \arrow[d, "T"] \\ W \otimes C_\bullet(X) \arrow[r, "\phi"] & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd} \tag{18}$$

is strictly commutative and $W \otimes C_\bullet(X)$ is equivalent to $C_\bullet(X)$ because W is contractible.

Remark 4.10. Let T act on W by composition, then we have an $\mathbb{F}_2$ action on W. Now we define $\mathbb{F}_2$ action on $W \otimes C_\bullet(X)$ by

$$T(x \otimes y) = (Tx) \otimes y$$

In this way, ϕ is a $\mathbb{F}_2$-equivalent map i.e. $T\phi = \phi T$. Hence it is a realization of an A-shaped diagram, where the shape of A is

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \bullet \arrow[loop right] \end{tikzcd}$$

We may write the diagram 14 into

$$\begin{tikzcd} C_\bullet(X) \arrow[r, "D_0"] & C_\bullet(X) \otimes C_\bullet(X) \arrow[loop right, "T"] \end{tikzcd} \tag{19}$$

where the left automorphism of A is mapped to the identity of $C_\bullet(X)$.

Remark 4.11 (The geometric realization of W). Let's consider the cellular structure of the infinity sphere:

$$S^\infty = \bigcup_{n=0}^{\infty} S^n$$

where $S^n \hookrightarrow S^{n+1}$ as an equator of S^{n+1}. Note that $S^{n+1} = D^{n+1} \cup S^n \cup TD^{n+1}$, where D^{n+1} is an $n+1$-cell as a hemisphere of S^{n+1} with boundary S^n and TD^{n+1} is the other one.

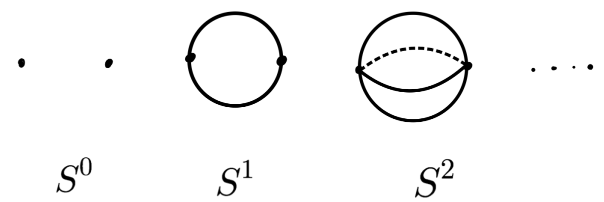

Figure 1. S^∞

Then we let $C_n(S^\bullet)$ be a free group of rank 2 with a basis D^n, TD^n and set

$$\begin{cases} \partial D^{2n} = D^{2n-1} + TD^{2n-1}, & \partial TD^{2n} = TD^{2n-1} + D^{2n-1} \\ \partial D^{2n+1} = TD^{2n} - TD^{2n}, & \partial TD^{2n+1} = D^{2n} - TD^{2n} \end{cases}$$

In this way, by $D^i \mapsto D_i$, we have $C_\bullet(S^\infty) = W$. Therefore, we have the diagram

$$\begin{tikzcd} S^\infty \times X \arrow[r, "\phi"] \arrow[d, "T\times \mathrm{id}"'] & X \times X \arrow[d, "T"] \\ S^\infty \times X \arrow[r, "\phi"] & X \times X \end{tikzcd} \tag{20}$$

We may view $S^1 \times X \to X \times$ as the diagram

$$\begin{tikzcd}[column sep=huge] C_\bullet(X) \arrow[r, bend left, "D_0"] \arrow[r, bend right, "TD_0"'] & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd}$$

$D^1 \times X \to X \times X$, the 2-cell filling the circle, as a homotopy

$$\begin{tikzcd}[column sep=huge] C_\bullet(X) \arrow[r, bend left, "D_0"] \arrow[r, bend right, "TD_0"'] \arrow[r, phantom, "D_1 \Downarrow"] & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd}$$

and similarly, TD^1 is another cell filling the circle

$$\begin{tikzcd} C_\bullet(X) \arrow[rr, bend left, "D_0"] \arrow[rr, bend right, "TD_0"'] \arrow[rr, phantom, "\Uparrow TD_1"] & & C_\bullet(X) \otimes C_\bullet(X) \end{tikzcd}$$

In this way, higher cells exhibits higher homotopies.

Recall the definition of cup product: for $u \in C^p(X)$ and $v \in C^q(X)$, the cup product is defined by

$$u \smile v \cdot c = u \otimes v \cdot D_0(c)$$

Similarly, we can define:

Definition 4.12. For each $i \geq 0$, we define a product called **cup-i product** as follows. for $u \in C^p(X)$ and $v \in C^q(X)$, the cup product is defined by

$$u \smile_i v \cdot c = u \otimes v \cdot \phi(D_i \otimes c)$$

for $c \in C_{p+q-i}(X)$.

Remark 4.13. The diagram 14 just gives cup products while the diagram 18 provides cup-i products. The cup-i products measure the deviation from the symmetry.

Proposition 4.3 (Differential formula)**.**

$$d(u \smile_i v) = u \smile_{i-1} v + v \smile_{i-1} u + du \smile_{i-1} v + u \smile_i dv \mod 2$$

Proof.

$$\begin{aligned} d(u \smile_i v)(c) &= (u \smile_i v)(\partial c) = u \otimes v(D_i(\partial c)) \\ &= u \otimes v[TD_{i-1}(c) + D_{i-1}(c) + \partial D_i(c)] \\ &= v \otimes u(D_{i-1}(c)) + u \otimes v(D_{i-1}(c)) + u \otimes v(\partial D_i(c)) \\ &= v \smile_{i-1} u + u \smile_{i-1} v + du \smile_i v + u \smile_i dv \end{aligned}$$

□

5. A geometric interpretation of the cup-i products on simplicial complexes

In this section, I will show the pictures of cup-i products to provide some intuition.

Suppose K is a simplicial complexes, then we we can endow K with a partial order of its vertices such that the vertices of any simplex are simply ordered. Let $V_0 < \cdots < V_n$ be a totally ordered subset of its vertices, then $[V_0, \ldots, V_n]$ is a coordinate of an n-simplex of K. However, there is more than one way to endow K with such a partial order, we just take one of them, say κ. Actually, what we need is independent of the choice of the orders. From now on, we always assume a simplicial K is ordered by κ and if we say there is a simplex $[V_0, \ldots, V_n]$ in K, it always means $V_0 < \cdots < V_n$ according to the partial order κ on the vertices of K.

Recall the definition cup products on simplicial complexes, suppose $\psi \in C^p(K)$ and $\varphi \in C^q(K)$, then

$$\psi \smile \varphi([V_0, \ldots, V_p, V_{p+1}, \ldots, V_{p+q}]) = \psi([V_0, \ldots, V_p])\varphi([V_{p+1}, \ldots, V_{p+q}])$$

For more intuition, we want to describe the cup products on the simplicial chain level instead of cochain level. Suppose σ is an n-simplex of K, then the dual of it σ^* is defined by

$$\sigma^*\colon C^n(K) \to \mathbb{Z},\ \text{an } n\text{-simplex } \tau \mapsto \begin{cases} 0 & \tau \neq \sigma; \\ 1 & \tau = \sigma; \end{cases}$$

This map $\sigma \mapsto \sigma^*$ induces an isomorphism $C_n(K) \to C^n(K)$. Note that the cup product define a bilinear map

$$\begin{array}{rrcl} \smile\colon & C^n(K) \times C^m(K) & \longrightarrow & C^{n+m}(K) \\ & \psi \times \varphi & \longmapsto & \psi \smile \varphi \end{array}$$

and by the composition with the isomorphisms, we have

$$\begin{array}{rrcl} \boxtimes: & C_n(K) \times C_m(K) & \longrightarrow & C_{n+m}(K) \\ & \sigma \times \tau & \longmapsto & \sum_{\alpha}^{\text{all } n+m\text{-simplexes}} (\sigma^* \smile \tau^* \cdot \alpha)\alpha \end{array} \tag{21}$$

where σ, τ are simplexes.

Example 5.1. Let $\sigma = [U_0, \ldots, U_n], \tau = [W_0, \ldots, W_m]$ be two simplexes in K, then

$$\sigma^* \smile \tau^* \cdot [V_0, \ldots, V_{n+m}] = (\sigma^* \cdot [V_0, \ldots, V_n])(\tau^* \cdot [V_n, \ldots, V_{n+m}])$$

Hence $\sigma^* \smile \tau^* = 0$ if $U_n \neq W_0$. If $U_n = W_0$, then

$$\sigma^* \smile \tau^* = [U_0, \ldots, U_n, W_1, \ldots, W_m]^*$$

and thus $\sigma \boxtimes \tau = [U_0, \ldots, U_n, W_1, \ldots, W_m]$.

Definition 5.2. Suppose σ, τ are simplexes of K, we say the ordered pair (σ, τ) are **0-regular pair** in the order κ if σ and τ has one vertex V in common and V is the last vertex of σ and the first vertex of τ. Namely, $([U_0, \ldots, U_n], [W_0, \ldots, W_m])$ is a 0-regular pair, if and only if $U_n = W_0$.

Note that the cup product is cup-0 product actually and the cup-0 product between two simplexes are non-trivial if and only if these two simplexes intersect on a 0-simplex in a regular position. To generalize the case, we need to define what the regular position of an i-simplex as the intersection of two simplexes is and try to use it to describe cup-i products.

Definition 5.3. Let σ, τ be two simplexes with dimension n, m and let i be a non-negative integer. The ordered pair (σ, τ) is ***i*-regular** in the order κ if the following conditions are satisfied

1. The vertices of σ, τ span a $(n+m-i)$-simplex ζ. In this case, σ, τ has $i+1$ vertices in common, denoted by $V_0, \ldots, V_i$ in the order κ.
2. V_0 is the first vertex of τ;
3. V_0, V_1 are adjacent vertices in σ;
4. V_1, V_2 are adjacent vertices in τ;
5. V_j, V_{j+1} are adjacent vertices in σ (resp. τ) if j is even (resp. odd) for all reasonable j;
6. V_i is the last vertex of σ (resp. τ) if i is even (resp. odd);

In particular, when $i = 0$, it coincides with Definition 5.2.

Definition 5.4 (The cup-i product on the level of chain complexes)**.** Suppose (σ, τ) is an i-regular pair in the order of κ, let σ_0 be the face of σ spanned by its vertices $\leq V_0$ and let σ_{2j} be the face of σ spanned by its vertices between V_{2j-1} and V_{2j} for $0 < 2j \leq i$, and if i is odd, let σ_{i+1} be the face spanned by its vertices $\geq V_i$. Similarly, let τ_{2j+1} be the face of τ spanned by its vertices between V_{2j} and V_{2j+1}, and if i is even, let τ_{i+1} be the face of τ spanned by its vertices $\geq V_i$. By the i-regularity, we have

$$\sigma = \sigma_0 \boxtimes \sigma_2 \boxtimes \cdots \boxtimes \sigma_{2k}$$

and

$$\tau = \tau_1 \boxtimes \tau_3 \boxtimes \cdots \boxtimes \tau_{2k+(-1)^i}$$

where $2k = i$ if i is even, and $2k = i + 1$ if i is odd.

Still by the i-regularity, $(\sigma_{2j}, \tau_{2j+1})$ and $(\tau_{2j+1}, \sigma_{2j+2})$ are 0-regular in the order κ and

$$\zeta = \sigma_0 \boxtimes \tau_1 \boxtimes \sigma_2 \boxtimes \tau_3 \cdots \boxtimes \begin{cases} \tau_{i+1} & i \text{ odd;} \\ \sigma_{i+1} & i \text{ even.} \end{cases}$$

Then we define the cup-i product on the level of chain level by

$$\sigma \smile_i \tau = \begin{cases} \zeta & (\sigma, \tau) \text{ is } i\text{-regular;} \\ 0 & \text{otherwise.} \end{cases}$$

The cup-i product on the cochain level is given by the isomorphisms $\sigma \mapsto \sigma^*$. Namely, for $u \in C^n(K), v \in C^m(K)$, we may write them into

$$u = \sum a_j \sigma_j^*, \ \ v = \sum b_k \tau_k^*$$

where σ_i are n-simplexes of K and τ_k are m-simplexes K, then

$$u \smile_i v = \sum_{j,k} a_j b_k (\sigma_j \smile_i \tau_k)^*$$

Example 5.5. Let K be a ordered simplicial set and there are four vertices $V_0 < V_1 < V_2 < V_3$, clearly, $([V_0, V_1, V_2], [V_1, V_2, V_3])$ is an 1-regular pair, the picture of the cup-1 product is $[V_0, V_1, V_2, V_3]$, see Figure 2.

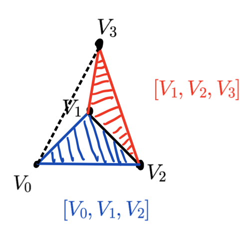

FIGURE 2. $[V_0, V_1, V_2] \smile_1 [V_1, V_2, V_3] = [V_0, V_1, V_2, V_3]$

Remark 5.6. Note that the Alexander-Whitney approximation D_0 12 actually provides us a way to decomposition an n-simplex into a sum of 0-regular pairs. When $i = 0$, the cup-0 product coincides with the cup product

$$(u \smile_0 v) \cdot [V_0, \ldots, V_{n+m}] = (u \cdot [V_0, \ldots, V_n])(v \cdot [V_n, \ldots, V_{n+m}])$$

When $i = 1$,

$$\begin{aligned}(u \smile_1 v)\cdot[V_0, \ldots, V_{n+m-1}]\\ &= \sum_{j=0}^{p-1} (u \cdot [V_0, \ldots, V_j, V_{j+m}, \ldots, V_{n+m-1}])(v \cdot [V_j, \ldots, V_{j+m}])\end{aligned}$$

In general, given a $p+q-i$-simplex α, any i-face of α determines a splitting of α into an i-regular pair (σ, τ) such that $(\sigma \smile_i \tau) = \pm\alpha$.

Theorem 5.7. *Let K, K' be two simplicial complex, and if $f : K \to K'$ an order preserving simplicial map, then $f^*(u \smile_i v) = f^*u \smile_i f^*v$.*

Theorem 5.8. *If u, v are cochains of dimensions p, q, then $d(u \smile_i v) = (-1)^{p+q-i} u \smile_{i-1} v + (-1)^{pq+p+q} v \smile_{i-1} u + du \smile_i v + (-1)^p u \smile_i dv$.*

The proof of these two theorem can be found in [Ste47].

Note that both the definition of cup-i products in Section 4 depends on the choice of ϕ and the definition in this section depends on the order α on the simplicial complex. To make them be independent of the choice, we need to pass it to cohomology.

6. The Steenrod squares and their properties

In this section, the definition of cup-i products follows Section 4.

By convention, we set $u \smile_{-1} v = 0$. If $u = v$ and $du = 0 \mod 2$, then $u \smile_i u$ is a cocycle modulo 2. Passing to cohomology classes gives a function

$$\begin{array}{rccc} Sq_i : & \tilde{H}^p(X; \mathbb{F}_2) & \longrightarrow & \tilde{H}^{2p-i}(X; \mathbb{F}_2) \\ & u & \longmapsto & u \smile_i u \end{array}$$

By setting $Sq^j := Sq_{p-j}$, one have

$$Sq^j : \tilde{H}^p(X; \mathbb{F}_2) \to \tilde{H}^{p+j}(X; \mathbb{F}_2)$$

These functions are called **Steenrod squares**.

Remark 6.1. Notice that cup-i products depend on the choice of ϕ. However, any two such ϕ are $\mathbb{F}_2$-equivalently homotopic. Thus the Steenrod squares are independent of the choice of ϕ.

In this subsection, we may assume all the cochain complexes are $\mathbb{F}_2$-coefficient.

Proposition 6.1. *The following statements are true:*

1. *If f is a continuous map, then $f^* Sq^i = Sq^i f^*$.*
2. *Sq^i is a group homomorphism.*
3. *$Sq^0 = \mathrm{id}$.*
4. *$Sq^n(u) = u \smile u$, if u is of n dimension.*
5. *$Sq^i(u) = 0$*

Proof. According to the definition of the cup_i products and the Steenrod squares, 3, 4 and 5 are straightforward. We now prove the rest.

(1) Consider the diagram

$$\begin{tikzcd} W \times X \arrow[r, "\phi_X"] \arrow[d, "\mathrm{id}\times f"'] & X \times X \arrow[d, "f \times f"] \\ W \times Y \arrow[r, "\phi_Y"] & Y \times Y \end{tikzcd}$$

and the diagram commutes up to homotopy because the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{D_X} & X \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow[D_Y]{} & Y \times Y \end{array}$$

then at level of cohomology, we have the following equations

$$\begin{aligned} f^*Sq_i(u)(c) &= f^*(u \smile_i u)(c) \\ &= f^*(u \otimes u)\phi_X(D^i \otimes c) \\ &= (f^*u \otimes f^*u)\phi_X(D^i \otimes c) \\ &= Sq_i f_*(u) \end{aligned}$$

(2) For any cocycle c and note that ϕ is T-equivalent, then we have

$$\begin{aligned} Sq_i(u+v)(v) &= (u+v) \smile_i (u+v)(c) \\ &= (u+v) \otimes (u+v)\phi(D^i \otimes c) \\ &= (u \otimes u + u \otimes v + v \otimes u + v \otimes v)\phi(D^i \otimes c) \\ &= Sq_i(u)(c) + Sq_i(v)(c) + u \otimes v(D_i(c)) + v \otimes u(D_i(c)) \\ &= Sq_i(u)(c) + Sq_i(v)(c) + u \otimes v(D_i(c)) + u \otimes v(TD_i(c)) \end{aligned}$$

Since $TD_i \simeq D_i$, then $D_i^*(u \otimes v) = TD_i^*(v \otimes u)$ in the cohomology group. Thus

$$Sq_i(u+v)(c) = Sq_i(u)(c) + Sq_i(v)(c) \mod 2$$

□

Now we define the cup_i product on the relative cohomology group: suppose $L \subseteq K$ as a subcomplex, then we have a short exact sequence

$$0 \longrightarrow C^\bullet(K,L) \xrightarrow{q^*} C^\bullet(K) \xrightarrow{j^*} C^\bullet(L) \longrightarrow 0$$

We may assume $\phi_L = \phi_K|_{W\otimes L}$, since $\phi_K(d_i \otimes \sigma) \in C(\sigma)$. Then for $u, v \in C^\bullet(K)$, $j^*(u \smile_i v) = j^*u \smile_i j^*v$. Let $u, v \in C^*(K,L)$, then $j^*(q^*u \smile_i q^*v) = 0$, then by the exactness, there is a unique $u \smile_i v \in C^*(K,L)$ such that $q^*(u \smile_i v) = q^*u \smile_i q^*v$ so that we can define cup_i products on the relative cochain in this way.

Proposition 6.2. *Suppose $L \subseteq K$ is a subcomplex and $\delta\colon \tilde{H}^n(L;\mathbb{F}_2) \to \tilde{H}^{n+1}(K,L)$ is the coboundary map, then $\delta Sq^i = Sq^i\delta$.*

Proof. Recall the definition of δ: Let a be an n-cocycle and $[a]$ be its cohomology class, then $[a] = j^*([b])$ for some $b \in C^n(K)$. Then $j^* \circ d(b) = d(a) = 0$, so $d([b]) = q^*([c])$ for some $c \in C^{n+1}(K,L)$.

Then we consider the diagram

$$\begin{array}{ccc} H^n(L) & \xrightarrow{Sq^i} & H^{n+i}(L) \\ {\scriptstyle \delta^*}\downarrow & & \downarrow{\scriptstyle \delta^*} \\ H^{n+1}(K,L) & \xrightarrow{Sq^i} & H^{n+i+1}(L) \end{array}$$

By the definition, $Sq^i(\delta^*[a]) = Sq^i(\bar{c}) = [c \smile_{n+1-i} c]$ and $Sq^i[a] = [a \smile_{n-i} a]$. We just need to show $\delta^*[a \smile_{n-i} a] = [c \smile_{n+1-i} c]$:

$$\begin{aligned} q^*[c \smile_{n+1-i} c] &= [q^*c \smile_{n+1-i} q^*c] \\ &= db \smile_{n+1-i} db \\ &= db' \end{aligned}$$

where $b' = b \smile_{n+1-i} db + b \smile_{n-i} b$ and the Equation (3.6) comes from the differential formula 4.3. Further, $j^*b' = j^*(b \smile_{n-i} b) = a \smile_{n-i} a$. As a result, $\delta^* Sq^i[a] = Sq^i \delta^*[a]$. □

By considering the pair (CX, X), where CX is the cone of X, we have the following corollary.

Corollary 6.1. *The Steenrod squares are stable, namely*

$$\Sigma Sq^i = Sq^i \Sigma$$

Theorem 6.2 (Cartan's formula)**.** *Let K and L be two complexes and for any two cohomology class $u \in H^*(K)$ and $v \in H^*(L)$, we have*

$$Sq^i(u \times v) = \sum_{p+q=i} Sq^p(u) \times Sq^q(v)$$

*which is called **Cartan's formula**.*

Proof. To prove the theorem, we first define a T equivariant map by

$$\begin{array}{rlcl} r: & W & \longrightarrow & W \otimes W \\ & D_i & \longmapsto & \sum_{0 \le j \le i} (-1)^{j(i-j)} D_j \otimes T D_{i-j} \end{array}$$

Let ϕ_K and ϕ_L be the chain maps inducing cup-i products on K and L respectively, then we consider the composition

$$W \otimes K \otimes L \xrightarrow{r \otimes \mathrm{id}} W \otimes W \otimes K \otimes L \xrightarrow{T} W \otimes K \otimes W \otimes L$$

$$\xrightarrow{\phi_K \otimes \phi_L} K \otimes K \otimes L \otimes L \xrightarrow{T} K \otimes L \otimes K \otimes L$$

where T is a suitable shuffle map. We claim that this map is homotopic to $\phi_{K \otimes L}$, since they are both carried by the same acyclic carrier clearly. Let $p = \dim u$, $q = \dim v$ and $n = p + q - i$. Thus

$$\begin{aligned} Sq^i(u \times v) \cdot (a \otimes b) &= ((u \otimes v) \smile_n (u \otimes v)) \cdot (a \otimes b) \\ &= (u \otimes v \otimes u \otimes v) \cdot \phi_{K \otimes L}(D_n \otimes a \otimes b) \\ &= (u \otimes u \otimes v \otimes v) \cdot \sum \phi_K(D_i \otimes a) \otimes T^j \phi_L(D_{n-j} \otimes b) \\ &= \sum (u \smile_j u \cdot a) \otimes (v \smile_{n-j} v \cdot b) \\ &= \sum Sq^{p-j}(a) \otimes Sq^{q-n+j}(b) \\ &= \sum (Sq^{p-j} u \times Sq^{q-n+j} v) \cdot (a \otimes b) \end{aligned}$$

Hence, since $Sq^i x$ is zero for i for $i > \dim x$, we have

$$\begin{aligned} Sq^i(u \times v) &= \sum_{j=0}^{n} Sq^{p-j} u \times Sq^{q-n+j} v \\ &= \sum_{s=i-q}^{p} Sq^s u \times Sq^{i-s} v \qquad s = p - j \\ &= \sum_{s=0}^{i} Sq^s u \times Sq^{i-s} v \end{aligned}$$

□

Corollary 6.2.
$$Sq^i(u \smile v) = \sum_{p+q=i} Sq^p(u) \smile Sq^q(v)$$

Thus we have the following morphism

Definition 6.3. Suppose X is a cell complex, then there is commutative graded ring homomorphism

$$\begin{array}{rccc} Sq: & H^*(X;\mathbb{F}_2) & \longrightarrow & H^*(X;\mathbb{F}_2)[t] \\ & x & \longmapsto & \sum Sq^i(x)t^i \end{array}$$

Remark 6.4. We consider the realization of the digram 13 in $\mathbb{S}$pace:

$$\begin{tikzcd} S^\infty \times X \arrow[r, "\phi"] \arrow[d, "T\times\mathrm{id}"'] & X \times X \arrow[d, "T"] \\ S^\infty \times X \arrow[r, "\phi"] & X \times X \end{tikzcd} \tag{22}$$

We quotient both sides by the group action, then we have

$$\tilde{\varphi}\colon S^\infty \times X/\mathbb{F}_2 \to X$$

where the $X \cong X \times X/\sim$ naturally. When passing to cohomology ring of coefficient $\mathbb{F}_2$, $\tilde{\varphi}^* = Sq$

$$Sq = \tilde{\varphi}^*\colon H^*(X;\mathbb{F}_2) \to H^*(X;\mathbb{F}_2)[t]$$

where $H^*(S^\infty \times X/\mathbb{F}_2;\mathbb{F}_2) = H^*(X;\mathbb{F}_2) \otimes H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = H^*(X;\mathbb{F}_2)[t]$ and t is a generator of $H^1(\mathbb{RP}^\infty;\mathbb{F}_2)$.

In [Ste47], by using the construction of cup-i products in Section 5, the induced Steenrod squares has the same properties in this section. In next section, we will see these properties determines Steenrod squares uniquely.

7. The uniqueness and existence of the Steenrod squares

Recall the representability of ordinary cohomology in Section 7, there is a canonical isomorphism

$$[X, K(\pi, n)] \cong H^n(X;\pi)$$

Theorem 7.1 (Classification theorem of cohomology operations)**.** *There is a canonical isomorphism*

$$\begin{array}{rccc} \vartheta: & \mathcal{O}(n,\pi;m,G) & \longrightarrow & H^m(K(\pi,n);G) \\ & \phi & \longmapsto & \phi(e_n) \end{array}$$

where e_n is the fundamental class of $H^n(K(\pi,n);\pi)$ corresponding to

$$[\mathrm{id}] \in [K(\pi,n), K(\pi,n)].$$

Sketch proof. The canonical isomorphism is given by

$$h \mapsto h_{K(\pi,n)}(F_\pi)$$

where h is cohomology operation and F_π is the fundamental class in $H^n(K(\pi,n);\pi)$. By Yoneda lemma, this is a canonical isomorphism. □

This definition makes sense, due the following theorem.

Theorem 7.2 (Existence and Uniqueness)**.** *For each $i \in \mathbb{N}$, the stable operations Sq^i satisfying the axioms in 1.6 exist and are unique.*

Proof. To determine such stable operations Sq^i, we just need to determine $Sq^i(e_n)$ for each i (e_n is the fundamental class in $H^n(K(\mathbb{F}_2,n);\mathbb{F}_2)$), because of the classification theorem.

First, we fix $n=1$, then we may define $Sq^0e_1 = e_1$, $Sq^1e_1 = e_1^2$ and $Sq^ie_1 = 0$ for $i > 1$.

Now we may argue it by induction on n. Suppose we have already define Sq^ie_{n-1} for each $i \in \mathbb{N}$, then we consider the spectral sequence associated to the fibration $K(\mathbb{F}_2,n)^I \to K(\mathbb{F}_2,n)$ with fiber $\Omega K(\mathbb{F}_2,n) \simeq K(\mathbb{F}_2,n-1)$. Note that the path space $K(\mathbb{F}_2,n)^I$ are contractible.
Step 1: Claim that for $i < n-1$, we have

$$\begin{cases} E_{n+1}^{0,n+i-1} & = H^{n+i-1}(K(\mathbb{F}_2,n-1);\mathbb{F}_2), \\ E_{n+i}^{n+i,0} & = H^{n+i}(K(\mathbb{F}_2,n);\mathbb{F}_2) \end{cases}$$

According to Leray's theorem, we have

$$\begin{cases} E_2^{0,n+i-1} & = H^{n+i-1}(K(\mathbb{F}_2,n-1);\mathbb{F}_2), \\ E_2^{n+i,0} & = H^{n+i}(K(\mathbb{F}_2,n);\mathbb{F}_2). \end{cases}$$

and the transgression

$$d_2^{0,n+i-1}\colon E_2^{0,n+i-1} \longrightarrow E_2^{2,n+i-2} = H^2(K(\mathbb{F}_2,n), H^{n+i-2}(K(\mathbb{F}_2,n)) = 0$$

because by Huriwicz's theorem, $H^i(K(\mathbb{F}_2,n) = 0$ for $i < n$. We now argue by induction to show

$$E_2^{0,n+i-1} = \cdots = E_{n+i}^{0,n+i-1}$$

for $i < n-1$.

We consider the transgressions again

$$d_k^{0,n+i-1}\colon E_k^{0,n+i-1} \longrightarrow E_k^{k,n+i-k},\ k < n+i$$

then either $k < n$ or $n+i-k < n-1$, because when $k \geq n$, we have $n+i-k \leq i < n-1$. Then such

$$E_2^{k,n+i-k} = H^k(K(\mathbb{F}_2,n); H^{n+i-k}(K(\mathbb{F}_2,n-1);\mathbb{F}_2)) = 0$$

because either $H^k(K(\mathbb{F}_2,n)) = 0$ or $H^{n+i-k}(K(\mathbb{F}_2,n-1)) = 0$ by Huriwicz's theorem. Then $E_k^{k,n+i-k} = 0$ and inductively, we can finish the step 1.

Step 2: Claim that for $i < n-1$, the transgressions

$$d_{n+i}^{0,n+i-1} \colon E_{n+i}^{0,n+i-1} \longrightarrow E_{n+i}^{n+i,0}$$

are isomorphisms.

According to the transgression theorem, the transgression is given the composition

$$H^{n+i-1}(F;\mathbb{F}_2) \xrightarrow{\delta^*} H^{n+i-1}(E,F;\mathbb{F}_2) \overset{(p^*)^{-1}}{\dashrightarrow} H^{n+i}(B,*;\mathbb{F}_2)$$

where $F = K(\mathbb{F}_2, n-1)$, $E = K(\mathbb{F}_2, n)^I \simeq *$, $B = K(\mathbb{F}_2, n)$. Since E is contractible, δ^* is an isomorphism. For $p^* \colon H^{n+i}(B,*;\mathbb{F}_2) \longrightarrow H^{n+i-1}(E,F;\mathbb{F}_2)$, we just need to check that $p_* \colon H_{n+i-1}(E,F;\mathbb{F}_2) \longrightarrow H_{n+i}(B,*;\mathbb{F}_2)$ is an isomorphism. First, it is surjective due to the lifting property of the fibration. Then it is injective because E is contractible.

Step 3: We just let

$$Sq^i e_n := d_{n+i}^{0,n+i-1}(Sq^i e_{n-1}).$$

Hereby $Sq^i e_n$ is defined for $i < n-1$. We just set $Sq^n e_n = e^n$ and $Sq^i e_n = 0$ when $i > n$. It remains to define $Sq^{n-1}e_n$.

Step 4: We will define $Sq^{n-1}e_n$ in this step.

Observe that for the transgression

$$d_n^{0,2n-2} \colon E_{n+1}^{0,2n-2} \longrightarrow E_n^{n,n-1}$$

we have

$$d_n^{0,2n-2}(Sq^{n-1}e_{n-1}) = d_n^{0,2n-2}(e_{n-1}^2) = 2e_{n-1}d_n^{0,2n-2}(e_{n-1}) = 0$$

Hence $Sq^{n-1}e_{n-1}$ is an element in $E_{n+1}^{0,2n-2}$.

By the method in Step 2, we have

$$E_{n+1}^{0,2n-2} = \cdots = E_{2n-1}^{0,2n-2}$$

because $E_k^{k,2n-2-k+1} = 0$ for $k \geq n+1$. We take $d_{2n-1}^{0,2n-2}(Sq^{n-1}e_{n-1})$ as $Sq^{n-1}(e_n)$, by the transgress

$$d_{2n-1}^{0,2n-2} \colon E_{2n-1}^{0,2n-2} \longrightarrow E_{2n-1}^{2n-1,0} = H^{2n-1}(K(\mathbb{F}_2, n);\mathbb{F}_2)$$

Step 5: Claim: the multiplication on the cohomology ring of any suspension space is trivial.

We first to prove that the diagonal map $D \colon \Sigma X \longrightarrow \Sigma X \times \Sigma X$ is homotopic to a map $\Sigma X \longrightarrow \Sigma X \vee \Sigma X$. We assume $x_0 \in X$ is the based point and we may write $\Sigma X = C_0 X \cup C_1 X$ where $C_i X$ are the cones. Then we define the homotopy by the formula $h_t(x) = (\varphi_t(x), \psi_t(x))$ where $h_0 = D$, $\varphi_1(C_0 X) = x_0$ and $\psi_1(C_1 X) = x_0$. This can be done because $C_i X$ are always contractible. Then $h_1(\Sigma X) \subseteq \Sigma X \vee \Sigma X$, which means that any two cohomology classes of X can be identified with a cohomology classes on $\Sigma X \vee \Sigma X$ by restriction. Note that the cross product of any two cohomology classes of X of positive dimension has zero restriction to $\Sigma X \vee \Sigma X$. Hence, in the cohomology of ΣX, the cup product of any two classes of positive dimension is always trivial.

Step 6: We now show that these operations are stable, namely, we need to check that the maps induced by suspension Σ are

$$\begin{array}{rrcl} \Sigma_n^i: & H^{n+i}(K(\mathbb{F}_2,n);\mathbb{F}_2) & \longrightarrow & H^{n+i-1}(K(\mathbb{F}_2,n);\mathbb{F}_2) \\ & Sq^i e_n & \longmapsto & Sq^i e_{n-1} \end{array}$$

For $i > n$, both of them are 0.

For $i < n$, Σ_n^i is the inverse of the transgression. According to the above construction, we have the result.

For $i = n$, $Sq^n e_n = e_n^2$ and $Sq^n e_{n-1} = 0$. Then we have the diagram

$$\begin{tikzcd} H^{2n}(K(\mathbb{F}_2,n);\mathbb{F}_2) \arrow[rr, "\Sigma_n^i"] \arrow[dr, "i_n^*"'] & & H^{2n-1}(K(\mathbb{F}_2,n-1);\mathbb{F}_2) \\ & H^{2n}(\Sigma K(\mathbb{F}_2,n-1);\mathbb{F}_2) \arrow[ur, "d_{2n}^{0,2n}"'] & \end{tikzcd}$$

Note that $K(\mathbb{F}_2,n-1) = \Omega K(\mathbb{F}_2,n)$ and $\Sigma\Omega K(\mathbb{F}_2,n)$ is weakly homotopy equivalent to $K(\mathbb{F}_2,n)$ by i_n. Then e_n^2 will be mapped to 0 in $H^{2n}(\Sigma K(\mathbb{F}_2,n-1);\mathbb{F}_2)$ according to the conclusion in Step 5.

Step 7: Uniqueness: Further, if they are stable cohomology operations, $Sq^i e_n$ must be the image of $Sq^i e_{n-1}$ for each i,n. Hence our construction is the unique one.

Step 8: Cartan's formula. Assume $\alpha \in H^m(X;\mathbb{F}_2)$ and $\beta \in H^n(X;\mathbb{F}_2)$ for $m,n > 0$, then we have the following result immediately

$$Sq^i(\alpha\cdot\beta) = \begin{cases} 0, & i > m+n \\ (\alpha\cdot\beta)^2 = (Sq^m\alpha)\cdot(Sq^n\beta), & i = m+n. \end{cases}$$

Hence we just need prove the case for $i < m+n$.

We argue the Cartan's formula by induction on i. Suppose the Cartan's formula holds for $i > m+n-s$ and $s > 0$, then we prove the case $i = m+n-s$. It is equivalent to check

$$Sq^{m+n-s}(\alpha\times\beta) = \sum_{p+q=m+n-s} Sq^p(\alpha)\times Sq^q(\beta)$$

for cross product. We may replace $X \times Y$ by $X \wedge Y$.

According to Yoneda lemma and the fact that products are compatible with induced morphism, we just need to check the case that $X = K_m = K(\mathbb{F}_2,m)$ and $Y = K_n = K(\mathbb{F}_2,n)$, $\alpha = e_m$ and $\beta = e_n$. Then consider

$$\begin{tikzcd} & (\Sigma K_{m-1})\wedge K_n = \Sigma(K_{m-1}\wedge K_n) \arrow[dl, "i_m\wedge id"'] \\ K_m \wedge K_n & \\ & K_m \wedge (\Sigma K_{n-1}) = \Sigma(K_{m-1}\wedge K_n) \arrow[ul, "id\wedge i_n"] \end{tikzcd}$$

where i_m, i_n are induced by Σe_{m-1} and Σe_{n-1}. Then we have

$$\begin{tikzcd} & H^r(\Sigma K_{m-1} \wedge K_n;\mathbb{F}_2) \\ H^r(K_m \wedge K_n;\mathbb{F}_2) \arrow[ur, "(i_m\wedge id)^*"] \arrow[dr, "(id\wedge i_n)^*"'] & \\ & H^r(K_m \wedge \Sigma K_{n-1};\mathbb{F}_2) \end{tikzcd}$$

and

$$\begin{tikzcd} & H^r(K_{m-1} \wedge K_n;\mathbb{F}_2) \\ H^r(K_m \wedge K_n;\mathbb{F}_2) \arrow[ur, "\Sigma^{-1}(i_m\wedge id)^*"] \arrow[dr, "\Sigma^{-1}(id\wedge i_n)^*"'] & \\ & H^r(K_m \wedge K_{n-1};\mathbb{F}_2) \end{tikzcd} \tag{23}$$

Recall that $f_m : H^r(K_m;\mathbb{F}_2) \to H^r(K_{m-1};\mathbb{F}_2)$ is $\Sigma^{-1} \circ i_m$ and is the inverse of transgression for $r < 2m$. Then in the diagram 23, we have

$$\begin{tikzcd} & f_m(\alpha) \times \beta \\ \alpha \times \beta \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & \alpha \times f_n(\beta) \end{tikzcd}$$

In particular,

$$\begin{tikzcd} & e_{m-1} \times e_n \\ e_m \times e_n \arrow[ur, mapsto] \arrow[dr, mapsto] & \\ & e_m \times e_{n-1} \end{tikzcd}$$

Observe that if $r < 2(m+n)$ and $\alpha \times \beta \in H^r(K_m \times K_n;\mathbb{F}_2)$ with $f_m(\alpha) = 0$ and $f_n(\beta) = 0$, then $\alpha \times \beta = 0$, because if $\alpha \neq 0$ and $\beta \neq 0$, then $\dim \alpha \geq 2m$ and $\dim \beta \geq 2n$,which implies $\dim(\alpha \times \beta) \geq 2(m+n)$, contradiction.

Now we consider

$$Sq^{m+n-s}(e_m \times e_n) - \sum_{p+q=m+n-s} Sq^p e_m \times Sq^q e_n \in H^{2(m+n)-s}(K_m \times K_n;\mathbb{F}_2)$$

then put it in the diagram 23, we have

$$\begin{cases} Sq^{m+n-s}(e_{m-1} \times e_n) - \sum\limits_{p+q=m+n-s} Sq^p e_{m-1} \times Sq^q e_n = 0 \\ Sq^{m+n-s}(e_m \times e_{n-1}) - \sum\limits_{p+q=m+n-s} Sq^p e_m \times Sq^q e_{n-1} = 0 \end{cases}$$

by the inductive hypothesis. Then by the previous observation, we have

$$Sq^{m+n-s}(e_m \times e_n) - \sum_{p+q=m+n-s} Sq^p e_m \times Sq^q e_n = 0$$

□

More details can be found in [FF16].

Appendix 1: The theory of simplicial sets

Definition 7.3. Let n be a non-negative integer, then the datum of the category $[n]$ consists of

- The set of objects is $\{0, 1, 2, \ldots, n\}$,
- The morphism is defined by

$$\mathrm{Hom}_{[n]}(k, j) = \begin{cases} \emptyset & k > j \\ \{\leq\} & k \leq j \end{cases}$$

which is called **natural category of** n.

For intuition, now we draw the diagrams of some natural categories:
The whole diagram of $[0]$:

$$\bullet$$

The whole diagram of $[1]$:

$$\bullet \longrightarrow \bullet$$

The whole diagram of $[2]$:

$$\begin{tikzcd} & \bullet \arrow[dr] & \\ \bullet \arrow[ur] \arrow[rr] & & \bullet \end{tikzcd}$$

The whole diagram of $[3]$:

$$\begin{tikzcd} & \bullet & \\ & & \bullet \arrow[ul] \\ \bullet \arrow[uur] \arrow[urr] \arrow[rr] & & \bullet \arrow[uul] \arrow[u] \end{tikzcd}$$

For greater n, it is very inconvenient to draw the whole diagram of $[n]$. Instead, the folded diagram of $[n]$ can be easy to show as

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

By observation, the diagram of $[n]$ looks like the standard geometric simplex $|\Delta^n|$: **the objects are vertices, the morphisms are edges and the compositions fulfil them.**

Given $m, n \in \mathbb{N}$, a functor $\alpha : [m] \to [n]$ is a non-decreasing map i.e. $k \leq j$ implies $\alpha(k) \leq \alpha(j)$.

Definition 7.4 (Simplex category)**.** Let Δ denote simplex category, the objects of simplex category are $\{[n]\}_{n \in \mathbb{N} \cup \{0\}}$ and the morphisms are functors between natural categories.

Definition 7.5 (Simplicial objects)**.** Given a category CC, a simplicial object in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. Morphisms between simplicial sets are natural transformations. In particular, a simplicial set is an object in $\mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set})$ and we denote the category $\mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set})$ of simplicial sets by sSet.

Remark 7.6. Since Set is cocomplete, namely, then sSet is also cocomplete, namely, sSet has all small colimits.

Example 7.7. For $[n] \in \Delta$, a simplicial set Δ_n defined by

$$\Delta_n([m]) := \mathrm{Hom}_\Delta([m], [n])$$

is called the **standard n-simplex**.

By Yoneda lemma, for any simplicial set $X_\bullet$, there is a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta_n, X) \simeq X_n$$

Elements in X_n are called **n-simplices** of X.

The **boundary** $\partial\Delta_n$ is defined by

$$\partial\Delta_n([m]) := \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid f \text{ is not surjective}\}$$

For $0 \le k \le n$, **the k-horn** Λ_n^k is defined by

$$\Lambda_n^k([m]) := \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid f([m]) \cup \{k\} \neq [n]\}$$

Notation 7.8. There are two elementary morphism classes in Δ

$$\delta^i \colon [n-1] \hookrightarrow [n], \ \ j \mapsto \begin{cases} j & j < i \\ j+1 & j \ge i. \end{cases}$$

$$\sigma^i \colon [n+1] \twoheadrightarrow [n], \ \ k \mapsto \begin{cases} j & j \le i \\ j-1 & j > i. \end{cases}$$

For any simplicial set X, the **face map** $d_k \colon X_n \to X_{n-1}$ is $X(\delta^k)$ and **the degeneracy map** $s_k \colon X_n \to X_{n+1}$ is $X(\sigma_k)$. A Simplex in X_{n+1} is called **degenerated simplex** if it is an image of a degeneracy map.

We will set that these two kinds morphisms are the most essential ones in the simplex category, because these two kinds of morphisms generates the simplex category. To see this, we need the following two lemmas.

Lemma 7.9. *If $f \colon [n] \to [m]$ is not surjective, then there exists δ^i and f' such that*

$$\begin{tikzcd} {[n]} \arrow[r, "f"] \arrow[dr, dotted, "f'"'] & {[m]} \\ & {[m-1]} \arrow[u, "\delta^i"'] \end{tikzcd}$$

Lemma 7.10. *If $g \colon [n] \to [m]$ is not injective, then there exists σ^k and g' such that*

$$\begin{tikzcd} {[n]} \arrow[r, "g"] \arrow[d, dotted, "\sigma^k"'] & {[m]} \\ {[n-1]} \arrow[ur, "g'"'] & \end{tikzcd}$$

Proposition 7.1. *Any morphism in Δ is a composition of some δ^i and σ^k.*

Example 7.11 (The nerve of category)**.** Given a locally small category $\mathcal{C}$, the simplicial set $N(\mathcal{C})$ defined by

$$N(\mathcal{C})_n := \mathcal{F}\mathrm{un}([n], \mathcal{C})$$

which is essentially the set of n-composable morphisms in $\mathcal{C}$. The degeneracy map is given by composition and the face map is given by adding an identity morphism. In particular, $N([n]) \cong \Delta_n$.

Actually, $N\colon \mathrm{Cat} \to \mathrm{sSet}$ is a fully faithful functor.

Definition 7.12 (Cosimplicial object)**.** Given a category $\mathcal{C}$, the covariant functors $\Delta \to \mathcal{C}$ is called cosimplicial set. The category of cosimplicial sets is denoted by cSet.

Given a cosimplicial object Q in the category $\mathcal{C}$, there is an associated functor $\mathrm{Sing}_Q\colon \mathcal{C} \to \mathrm{sSet}$ defined by

$$\mathrm{Sing}_Q(X)_n := \mathrm{Hom}_{\mathcal{C}}(Q(\Delta_n), X)$$

Proposition 7.2. *If $\mathcal{C}$ is a cocomplete category, given then there is an adjunction*

$$(|-|_Q, \mathrm{Sing}_Q)\colon \mathcal{F}\mathrm{un}(\Delta^{\mathrm{op}}, \mathrm{Set}) \rightleftharpoons \mathcal{C}$$

In particular, let $S_\bullet$ be a simplicial set

$$|S_\bullet|_Q := \varinjlim_{\Delta/S_\bullet} p_Q$$

where $\Delta/S_\bullet$ is the ***simplex category of*** $S_\bullet$ *with objects $([n], \Delta_n \to S_\bullet)$ and*

$$p_Q\colon ([n], \Delta_n \to X) \mapsto Q([n])$$

Sketch proof. Use small object argument, see [Gro10] and [KS06]. □

Example 7.13 (Geometric realization and singular functor)**.** The n-standard simplex Δ_n is defined as follows

$$|\Delta_n| = \{(x_0, x_1, \cdots, x_n) \in I^n \mid \sum x_i = 1\}$$

Then $\Delta_n \to |\Delta_n|$ determines a cosimplicial space and determines an adjunctions

$$(|-|, \mathrm{Sing})\colon \mathrm{sSet} \rightleftharpoons \mathrm{Top}$$

For a simplicial set $S_\bullet$, the topological space (actually a simplicial/cellular complex) $|S_\bullet|$ is called **the geometric realization** of $S_\bullet$. In details,

$$|S_\bullet| = \coprod_{n \geq 0} |\Delta_n| \times S_n / \sim$$

where the relation $\sim$ is given by the face maps and the degeneracy maps. The original idea is in [Mil57]. Sing is called **singular functor**.

This adjunction is actually a Quillen equivalence, see [Qui67].

Example 7.14 (Chain complex realization and the normalize functor)**.** In this example, we will show an adjunction between the category of simplicial abelian groups and the category of chain complexes.

Given a simplicial abelian group $A_\bullet$, the normalized chain complex $\mathcal{N}_\bullet(A_\bullet)$ of $A_\bullet$ is defined by

$$\mathcal{N}_q(A_\bullet) = \bigcap_{i=0}^{q-1} \ker(d_i\colon A_q \to A_{q-1})$$

Given a simplicial set $S_\bullet$, the free generated simplicial abelian group $\mathbb{Z}[S_\bullet]$ is defined by

$$\mathbb{Z}[S_\bullet]_n := \mathbb{Z}[S_n], \text{the free abelian group generated by } S_n$$

Then we define a cosimplicial chain complex by

$$[n] \mapsto \Delta_n \mapsto \mathbb{Z}[\Delta_n] \mapsto \mathcal{N}_\bullet(\mathbb{Z}[\Delta_n])$$

and we denoted the adjunction associated to this cosimplicial chain complex by

$$(\mathcal{N}_\bullet, \mathbb{Z}[-], \Gamma) \colon \mathrm{sSet} \rightleftharpoons \mathrm{Ch}$$

Actually, $\mathcal{N}_\bullet$ is a natural equivalence (and a lax monoid functor), see [GJ09], which is called **Dold-Kan correspondence**.

There is another equivalent definition of normalized complex. Given a simplicial abelian group $A_\bullet$, **the Moore complex** $\mathcal{M}_\bullet(A_\bullet)$ associated to $A_\bullet$ is defined by

$$\mathcal{M}_n(A_\bullet) := A_n$$

and the boundary map

$$\partial_n := \sum_{i=0}^{n} d_i \colon A_n \to A_{n-1}$$

the degenerate complex $D_\bullet(A_\bullet)_n$ associated to $A_\bullet$ is a subcomplex of $\mathcal{M}_\bullet(A_\bullet)$defined by

$$D_n(A_\bullet) := \text{free abelian groups generated by degenerated } n\text{-simplices in } A_\bullet$$

Then there is a fact that $N_\bullet(A_\bullet) = \mathcal{M}_\bullet(A_\bullet)/D_\bullet(A_\bullet)$, see also [GJ09].

Thus we have the diagram

$$\begin{tikzcd}
 & \mathrm{sSet} \arrow[dl, shift left, "\mathcal{N}_\bullet"] \arrow[dr, shift left, "|-|"] & \\
\mathrm{Ch} \arrow[ur, shift left, "\Gamma"] & & \mathbb{S}\mathrm{pace} \arrow[ul, shift left, "\mathrm{Sing}"'] \arrow[ll, "C_\bullet(-;\mathbb{Z})"]
\end{tikzcd}$$

Moreover, the homotopy theory on these three categories are compatible in this diagram. In particular, the interval objects coincide

$$I = |\Delta_1| = \Delta_1 = I_\bullet = \mathcal{N}_\bullet(\mathbb{Z}[\Delta_1])$$

Appendix 2: Monoidal categories and enriched categories

Definition 7.15. Let $\mathcal{C}$ be a category. A **nonunital monoidal structure** on $\mathcal{C}$ consists of the following data:

(1) A functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which is called **tensor product functor**.
(2) A collection of isomorphisms $\alpha_{X,Y,Z} X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$, for any X, Y, Z in $\mathcal{C}$, called **associativity constraints** of $\mathcal{C}$.

and the data satisfies the following rules:

(1) for every triple of morphisms $f\colon X \to X'$, $g\colon Y \to Y'$, and $h\colon Z \to Z'$, the diagram

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "f\otimes(g\otimes h)"'] & (X \otimes Y) \otimes Z \arrow[d, "(f\otimes g)\otimes h"] \\ X' \otimes (Y' \otimes Z') \arrow[r, "\alpha_{X',Y',Z'}"'] & (X' \otimes Y') \otimes Z' \end{tikzcd}$$

commutes.

(2) For every quadruple of objects W, X, Y, Z in $\mathcal{C}$, the diagram of associativity constraints commutes

$$\begin{tikzcd} W \otimes ((X \otimes Y) \otimes Z) \arrow[r] & (W \otimes ((X \otimes Y)) \otimes Z \arrow[d] \\ W \otimes (X \otimes (Y \otimes Z)) \arrow[u] \arrow[d] & (W \otimes ((X \otimes Y)) \otimes Z \\ (W \otimes X) \otimes (Y \otimes Z) \arrow[ur] & \end{tikzcd}$$

Let $(\otimes, \alpha)$ denote the nonunital monoidal structure.

A **unit** in $(\otimes, \alpha)$ is a pair (e, v) where e is an object $\mathcal{C}$ and $v\colon e \otimes e \to e$ is an isomorphism, which satisfies the following condition: the functors

$$\mathcal{C} \longrightarrow \mathcal{C} \qquad C \longmapsto e \otimes C$$

and

$$\mathcal{C} \longrightarrow \mathcal{C} \qquad C \longmapsto C \otimes e$$

are fully faithful. Since these two functors are fully faithful, for each X in $\mathcal{C}$, we have **right unit constraints** $\lambda_X : e \otimes X \to X$ and **left unit constraints** $\rho_X\colon X \otimes e \to X$ induced by v and the associative constraints.

A monoidal category with unit object e is a category $\mathcal{C}$ with $(\otimes, \alpha, e, v)$.

Example 7.16. The categories Vect, Set, sSet,Ch, Mod, $\mathbb{S}$pace, Top have monoidal structures.

Definition 7.17 (Lax nonunital monoidal functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two monoidal categories, a lax nonunital monoidal functor $F\colon \mathcal{C} \to \mathcal{D}$ is a functor with a collection of isomorphisms $\mu_{X,Y}\colon F(X \otimes Y) \to F(X) \otimes F(Y)$ for each X, Y in $\mathcal{C}$ such that the following diagram commutes for any pair of morphisms $f\colon X \to X'$, $g\colon Y \to Y'$ in $\mathcal{C}$

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "\mu_{X,Y}"] \arrow[d, "F(f)\otimes F(g)"] & F(X \otimes Y) \arrow[d, "F(f\otimes g)"] \\ F(X') \otimes F(Y') \arrow[r, "\mu_{X',Y'}"] & F(X' \otimes Y”) \end{tikzcd}$$

and these morphisms are compatible with the associativity constraints on $\mathcal{C}$ and $\mathcal{D}$.

Example 7.18. The realization functor, the singular functor and Dold-Kan correspondence are all lax monoidal functors.

Definition 7.19 (Enriched categories)**.** Let $\mathcal{A}$ be a monoidal category with unit object e. An $\mathcal{A}$-enriched category $\mathcal{C}$ consists of the following data:

(1) A collection of objects;
(2) For every pair of objects X, Y in $\mathcal{C}$, there is an object $\mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y)$ in $\mathcal{A}$;
(3) For every triple of objects X, Y, Z in $\mathcal{C}$, there is a morphism

$$c_{X,Y,Z} \colon \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(Y,Z) \otimes \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Z)$$

in $\mathcal{A}$, which is called the composition law;
(4) For every object X in $\mathcal{C}$, there is a morphism $e_X \colon e \to \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,X)$ as the identity of X;

and these data should satisfy the following rules:
(1) the composition law is associative;
(2) for any objects X, Y, the following diagrams commute

$$\begin{tikzcd} e \otimes \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \arrow[rr, "e_Y \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & & \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(Y,Y) \otimes \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \arrow[dl, "c"] \\ & \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \otimes e \arrow[rr, "\mathrm{id} \otimes e_X"] \arrow[dr, "\rho"'] & & \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,X) \arrow[dl, "c"] \\ & \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) & \end{tikzcd}$$

where λ, ρ, c are left unit constraints, right unit constraints and the composition law.

Given two $\mathcal{A}$-enriched category $\mathcal{C}$ and $\mathcal{D}$, an $\mathcal{A}$-enriched functor F consists of a collection morphisms $F_{X,Y} \colon \mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}^{\mathcal{A}}(X,Y)$ that preserve identities.

Remark 7.20. Given a lax monoidal functor $G \colon \mathcal{A} \to \mathcal{B}$ and an $\mathcal{A}$-enriched category $\mathcal{C}$, then there is a **change of base** to make $\mathcal{C}$ a $\mathcal{B}$-enriched category via G by

$$\mathrm{Hom}_{\mathcal{C}}^{\mathcal{B}}(X,Y) := G\mathrm{Hom}_{\mathcal{C}}^{\mathcal{A}}(X,Y)$$

Example 7.21. Given a commutative ring, the category of R-module Mod is enriched by itself. Similarly, by taking operator complexes in Definition 3.6, Ch is enriched by itself.

sSet is enriched by it self by setting

$$\mathrm{Hom}_{\mathrm{sSet}}^{\mathrm{sSet}}(X,Y)_n := \mathrm{Hom}_{\mathrm{sSet}}(X \times \Delta_n, Y)$$

for simplicial sets X, Y.

$\mathbb{S}$pace is a category enriched by Top, see next section.

Appendix 3: Compact-open topology for mapping spaces

Motivation 7.22. In the category Mod_R we have the typical adjunction:

$$\mathrm{Hom}(L \otimes M, N) \cong \mathrm{Hom}(L, \mathrm{Hom}(M,N))$$

The key point is that, in Mod, the morphism set has a module structure naturally. This is a good property. We wonder if we can do similar things on topological space. We want to define an appropriate topology on the set $\mathrm{Hom}_{\mathrm{Top}}(X,Y))$ so that the adjunction still works, or still works for some good topological space, for example locally compact, Hausdorff,etc. Moreover, for any topological space X, we want $\mathrm{Hom}_{\mathrm{Top}}(-, X)$ or $\mathrm{Hom}_{\mathrm{Top}}(X, -)$ to be functor from Top to Top.

Let $Y^X := \mathrm{Hom}_{\mathrm{Top}}(X, Y)$ be the set of continuous maps from topological space X to topological space Y. For $K \subset X$, $U \subset Y$, we denote

$$\mathcal{W}(K, U) := \{f \in Y^X : f(K) \subset U\}$$

Definition 7.23 (Compact-open topology)**.** The compact-open topology on Y^X is generated as subbasis by

$$\{\mathcal{W}(K, U) : \text{compact } K \subset X, \text{open } U \subset Y\}$$

The following propositions will show why we endow topology on Y^X in this way.

Proposition 7.3. *Let $f : X \to Y$ be a continuous map, for any topological space Z, the induced maps $f^Z : Y^Z \to X^Z$ and $Z^f : Z^X \to Z^Y$ are continuous.*

Proof. Let K be a compact subset of X, U be an open set of Z, then $\mathcal{W}(K, U)$ is an open in X^Z be the definition. It suffices to show the preimage of $\mathcal{W}(K, U)$ is an open set in Y^Z. Since f is continuous, $f(K)$ is still compact in Y, then $\mathcal{W}(f(K), U)$ is open in Y^Z and $Z^f(\mathcal{W}(f(K), U)) \subset \mathcal{W}(K, U)$.

For Z^f, let L be a compact subset of Z, O be an open subset of Y, and we consider $Z^f(\mathcal{W}(L, f^{-1}(O)) \subset \mathcal{W}(L, O)$, then the result is straightforward. □

Remark 7.24. This proposition shows that why we choose compact sets and open sets to construct compact-open topology. The reason is that under a continuous map, compact sets are mapped to compact sets while the preimage of an open set is an open set. One may ask:'Why not use connected sets? Connected sets are still preserved under continuous maps.' The following proposition will show us why we prefer compact sets.

Proposition 7.4. *Let $f : X \times Y \to Z$ be a continuous map,* ***the adjoint map*** *$f^\wedge : X \to Z^Y$ by $f^\wedge(x)(y) := f(x, y)$ is continuous.*

Proof. For $\mathcal{W}(K, U) \subset Z^Y$, where K is compact and U is open, we want to show $f^{\wedge -1}(\mathcal{W}(K, U))$ is open. For any $x \in f^{\wedge -1}(\mathcal{W}(K, U))$, $f(\{x\} \times K) \subset U$ and f^{-1} is open and covers $\{x\} \times K$. By the definition of product topology, we may write $f^{-1}(U) = \bigcup_{i \in I} A_i \times B_i$ where $A_i \subset X$, $B_i \subset Y$ and both are open. Since K is compact, there exists a finite subcover $\bigcup_{n=1}^N A_n \times B_n$ to cover $\{x\} \times K$ and we may require $x \in A_n$ for each n. Let $V = \bigcap_{n=1}^N A_n$, which is open in X and we have $V \times K \subset f^{-1}(U)$ i.e. $f(V \times K) \subset U$ i.e. $f^\wedge(V) \subset \mathcal{W}(K, U)$. □

Remark 7.25. Clearly, in the category Set, we have a canonical bijection (actually a a pair of adjoint functors):

$$\mathrm{Hom}_{\mathrm{Set}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathrm{Set}}(X, \mathrm{Hom}_{\mathrm{Set}}(Y, Z))$$

Proposition 7.4 shows that, if we endow the morphism set with compact-open topology, the canonical map is well-defined in Top, which is the reason why we require compactness instead of connectedness when defining the topology! We need the finiteness of subcover!

We always assume that Y^X is endowed with compact-open topology when it is mentioned as a topological space. **The next question is whether the canonical**

map is a bijection with compact-open topology. To formulate this question more clearly, we define some notations:

Let X, Y, Z be topological spaces, then the **evaluation** $\mathrm{ev}_{X,Y} = \mathrm{ev}\colon Y^X \times X \to Y$ defined by $(f, x) \mapsto f(x)$.

Given a continuous map $g\colon X \to Z^Y$, define the map $g^\vee = \mathrm{ev}_{Y,Z} \circ (g \times \mathrm{id}_Y)$.

Define

$$\alpha\colon Z^{X\times Y} \to (Z^X)^Y$$

by setting $\theta(f) := f^\wedge$ and define

$$\beta\colon (Z^X)^Y \to Z^{X\times Y}$$

by setting $\beta(g) := g^\vee$.

Clearly, in Set, $\alpha \circ \beta = \mathrm{id}_{(Z^X)^Y}$ and $\beta \circ \alpha = \mathrm{id}_{Z^{X\times Y}}$. When it comes to Top, to answer the question, we first need to check β is well defined. The key point is whether the evaluation map is continuous.

Proposition 7.5. *Let X be a locally compact (in my notes, compactness requires Hausdorff while quasicompactness does not have to) topological space. Then the **evaluation** $\mathrm{ev}_{X,Y} = \mathrm{ev}\colon Y^X \times X \to Y$ defined by $(f, x) \mapsto f(x)$ is continuous.*

Proof. Let U be an open neighbourhood of $f(x)$. Since f is continuous, $f^{-1}(U)$ is a neighbourhood of x. Recall the definition of locally compactness, we can find a compact nbhd K of x such that $K \subset f^{-1}(U)$ i.e. $f(K) \subset U$. Hence $\mathcal{W}(K, U) \times K \subset \mathrm{ev}^{-1}(U)$. $\square$

Corollary 7.1. *Suppose X, Y are locally compact, then for any topological space Z, we the the canonical bijection (homeomorphism actually):*

$$Z^{X\times Y} \cong (Z^X)^Y$$

We will see that locally compact spaces are good enough to have this good bijection. However, locally compactness is not the necessary condition. For more sophisticated description of the spaces that own the canonical bijection in Corollary 7.1, Steenrod gave us the suitable subcategory of Top is the category of compactly generated spaces, see in [Ste67]. Let $\mathbb{S}$pace be this good category that all the objects in $\mathbb{S}$pace admit the canonical bijection. We simply call these objects spaces. To study algebraic topology, we just need to focus on $\mathbb{S}$pace instead of Top. Let $\mathrm{Map}(X, Y)$ be the topological space that is $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$ with compact-open topology.

Definition 7.26. The homotopy category of spaces $\mathrm{Ho}(\mathbb{S}\mathrm{pace})$ has the same objects as $\mathbb{S}$pace, the morphisms are homotopy classes, i.e.

$$\mathrm{Hom}_{\mathrm{Ho}(\mathbb{S}\mathrm{pace})}(x, y) = \pi_0 \mathrm{Map}(x, y)$$

Moreover, in $\mathbb{S}$pace$*$, the evaluation maps are continuous and $(X, x_0)^{(Y, y_0)} := \mathrm{Map}((X, x_0), (Y, y_0))$ with compact-open topology (sometimes, we may omit the based point for convenience), does the bijection still holds?

Let $(X, x_0), (Y, y_0), (Z, z_0)$ be three pointed spaces, and the based point of Z^Y is the constant map from Y to z_0. If we require $\alpha : Z^{X\times Y} \to (Z^X)^Y$ is a based map, then $f^\wedge(x_0)$ is the constant map i.e. $f(x_0, Y) = z_0$. Switch the positions of X and Y, we have $f(X, y_0) = z_0$. Hence we have $f(\{x_0\} \times Y \cup X \times \{y_0\}) = z_0$.

Conversely, for a based map $\varphi\colon Y \to Z^X$ i.e. $\varphi(y_0)(x) = z_0, \forall x \in X$, we have $\varphi^\vee(X \times \{y_0\}) = z_0$. Similarly, switch the positions of X and Y again, we have $\varphi^\vee(X \times \{y_0\} \cup \{x_0\} \times Y) = z_0$. In summary, we have

$$\{f \in \mathrm{Map}(X \times Y, Z) : f(\{x_0\} \times Y \cup X \times \{y_0\}) = z_0\} \cong \mathrm{Map}(X, Z^Y) \tag{24}$$

Definition 7.27 (Smash product)**.** The **smash product** of two pointed topological spaces $(X, x_0), (Y, y_0)$ is

$$X \wedge Y := X \times Y / \{x_0\} \times Y \cup X \times \{y_0\}$$

Thus, according to Eq.(24), the adjunction in $\mathbb{S}$pace$*$ is given by

$$\mathrm{Map}(X \wedge Y, Z) \cong \mathrm{Map}(X, Z^Y) \tag{25}$$

Remark 7.28. Smash product is similar to tensor product $\otimes$. Observe that the based point of a pointed topological space functions like $\otimes$ of a module.

Appendix 4: Obstruction theory and representability theorems

Motivation 7.29. Let B be a CW-complex and $A \subseteq B$ be a subcomplex. Let $X^n = A \cup B^n$ and given a map $f : X^n \to Y$, where Y is an n-simple space i.e. $\pi_1(Y)$ acts on $\pi_n(Y)$ trivially i.e. $S^n \to Y$ determines an element in $\pi_n(Y)$ which is independent of the choice of based point. Now the question is: **Can we extend this map to B^{n+1}?** We may try to do it cell by cell. Let σ be an $n+1$-cell of B which is not in A and $\tilde{\sigma} : S^n \to X^n \subset B$ is the characteristic map, then $f \circ \tilde{\sigma}\colon S^n \to Y$ is an element in $\pi_n(Y)$.

Observation 1: f can be extended to $X^n \cup \sigma = X^n \cup_{\tilde{\sigma}} D^{n+1}$ if and only if $f \circ \tilde{\sigma}$ is null-homotopic.

Let $c(f)\colon \sigma \mapsto [f \circ \tilde{\sigma}]$ be a cochain in $C^{n+1}(B, A; \pi_n(Y))$ (recall the definition of cellular cohomology).

Observation 2: If $c(f) = 0$, then we can extend f to B^{n+1}.

We call $c(f)$ **the obstruction to extending f over B^{n+1}** or simply **obstruction cocycle**. (*We haven't check that $c(f)$ is a cocycle yet, but it is a fact.*)

Proposition 7.6. *Let K be a CW-complex of dimension $\leq n$ and X is an n-connected space, then any map $f\colon K \to X$ is null-homotopic.*

Proof. Take $(B, A) = (K \times I, K \times \partial I)$. Let $c\colon K \to X$ be a constant map, then we have $f \sqcup c\colon K \times \partial I \to X$. Note that X is n-connected, hence the obstruction of $f \sqcup c$ vanishes (actually, the whole cohomology group of coefficient $\pi_n(X)$ vanishes), hence we may extend $f \sqcup c$ to $K \times I$, which is the homotopy that we need. $\square$

Definition 7.30 (Difference cochain)**.** Let $f, g\colon X^n \to Y$ be two maps that agree on X^{n-1}. Given an n-cell σ not in A with the characteristic map $\tilde{\sigma} : S^{n-1} \to X^{n-1}$. Then we define $d(f, g)_\sigma : S^n \to Y$ by talking $S^n \cong D^n_+ \cup S^{n-1} \cup D^n_-$ with $d(f, g)_\sigma|_{D^n_+} = f|_\sigma$ and $d(f, g)_\sigma|_{D^n_-} = g|_\sigma$. Thus we get an n-cochain in $C^n(B, A; \pi_n(Y))$ by defining

$$d(f, g) : \sigma \mapsto [d(f, g)_\sigma]$$

which is called **difference cochain between f and g**.

Proposition 7.7. *Given $f : X \to Y$ and a cochain $d \in C^n(B, A; \pi_n(Y))$, then there exists $g\colon X^n \to Y$ such that $d(f, g) = d$ and $f|_{X^{n-1}} = g|_{X^{n-1}}$.*

Proof. We just need to deal with the simple case: given $d_\sigma\colon S^n \to Y$ and $f_\sigma\colon D^n \to Y$, there exists $g_\sigma\colon S^n \to Y$ such that $g_\sigma|_{D^n_+} = f_\sigma$ and $g_\sigma \simeq d_\sigma$.

Regard D^n as a subcomplex of S^n and clearly $d\sigma|^n_D\colon D^n \to Y$ is homotopic to $f_\sigma\colon D^n \to Y$. Let h e such homotopy, then by HEP for CW pair (S^n, D^n), we have a homotopy $H\colon S^n \times I \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
D^n \arrow[rr, "i_0^{D^n}"] \arrow[dd, "i"'] & & D^n \times I \arrow[dl, "h"'] \arrow[dd, "i\times\mathrm{id}"] \\
& Y & \\
S^n \arrow[ur, "d_\sigma"] \arrow[rr, "i_0^{S^n}"] & & S^n \times I \arrow[ul, dotted, "H"']
\end{tikzcd}$$

we just let $g_\sigma = h_1 = h(-,1)$ and it is done. □

Lemma 7.31. *There is a coboundary formula for the difference cochains:*

$$\delta d(f,g) = c(g) - c(f) \tag{26}$$

Proof. First, we consider the simplest non-trivial case where f and g are different on only one n-cell $e \subset X$. Let σ be an $(n+1)$-cell of X; we want to show that

$$c(g)\cdot\sigma - c(f)\cdot\sigma = [\sigma : e]d(f,g)\cdot e$$

where $[\sigma : e]$ is the coefficient of e in $\partial\sigma$.

Let $\phi\colon D^{n+1} \to X$ be a characteristic map for σ. We may assume that $\phi^{-1}(e)$ consists of several open balls, of which every one is mapped homeomorphically onto e, with preserving or reversing the orientation, and recall the definition of the boundary maps in the cellular chain complex, $[\sigma : e]$ is the difference of the number of balls where the orientation is preserved and the number of balls where the orientation is reversed. Then we represent $c(f)\cdot\sigma$ and $c(g)\cdot\sigma$ by the maps of spheres, then see the following picture

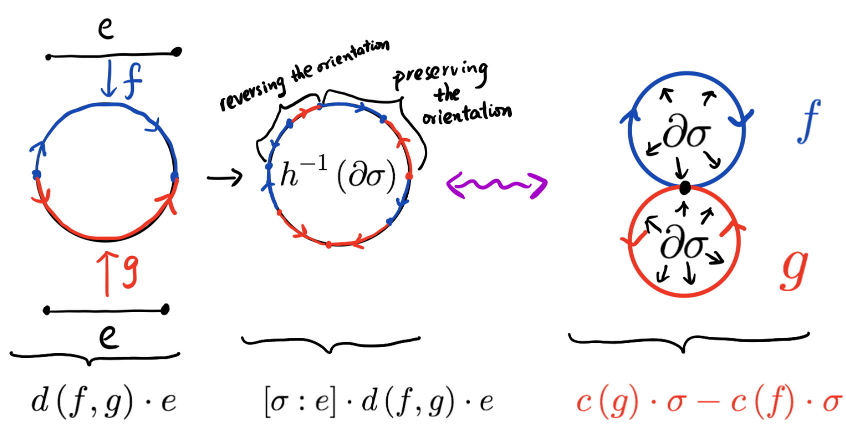

FIGURE 3. $c(g)\cdot\sigma - c(f)\cdot\sigma = [\sigma : e]d(f,g)\cdot e$

For the general case, we just check it cell by cell and the proof is completed. □

Lemma 7.32. *There is an addition formula for the difference cochains*

$$d(f,h) = d(f,g) + d(g,h) \tag{27}$$

Sketch proof. The proof is similar to the Figure 3. □

Theorem 7.33. *There is a map $g\colon X^{n+1} \to Y$ which agrees with f on X^n if and only if $[c(f)] = 0$.*

Proof. If $[c(f)] = 0$, then there exists $d \in C^n(B, A; \pi_n(Y))$ such that $c(f) = \delta(d)$. By previous proposition, we can find $g\colon X^n \to Y$ such that $g|_{X^{n-1}} = g|_{X^{n-1}}$ and $d(f,g) = -d$. Then $c(f) = \delta(d) = -\delta(d(f,g)) = c(f) - c(g) = 0$, then we can extend g to X^{n+1}.

Conversely, if there is such g, let $d = d(f, g|_{X^n})$ and $[c(f)] = 0$ by considering $\delta(d)$. □

Remark 7.34. The key point is to use cochain or cocycle to determine the existence of homotopy, by taking $(B, A) = (K \times I, K \times \partial I)$.

Theorem 7.35. *Let K be a CW-complex of dimension n $f|_{K^n} \simeq g|_{K^n}$ relative on K^{n-1} if and only if $d(f,g) = 0$ in $C^n(K;\pi_n(Y))$.*

$f|_{K^n} \simeq g|_{K^n}$ relative on K^{n-2} if and only if $[d(f,g)] = 0$ in $H^n(K;\pi_n(Y))$.

Sketch proof. If $f, g\colon K \to Y$ agree on K^{n-1}, we take $(B, A) = (K\times I, K\times\partial I)$, then there is a natural map $k\colon X^n \to Y$(recall that $X^n = A \cup B^n$) such that $k|_{K^n\times\{0\}} = f$, $k|_{K^n\times\{1\}} = g$ and $k|_{K^{n-1}\times I}(x,t) = f(x) = g(x)$. Further, $c(k)$ corresponds to $d(f|_{K^n}, g|_{K^n})$. Then if $d(f,g) = 0$ in $C^n(K;\pi_n(Y)) \cong C^{n+1}(\Sigma K;\pi_n(Y))$. Note that $C^{n+1}(B, A;\pi_n(Y)) \cong C^{n+1}(B/A;\pi_n(Y)) \cong C^{n+1}(\Sigma K;\pi_n(Y))$. Hence $c(k) = 0$ implies that we can extend $k\colon X^n \to Y$ to $H\colon K \times I \to Y$, which is the homotopy we need. Conversely, we just reverse the direction of previous argument to show it is also true.

Similarly, we may use Theorem 7.33 to show the second assertion. □

7.0.1. *The fundamental classes of Eilenberg-Maclane spaces.*

Definition 7.36. Let π be an abelian group and n be a positive integer, the Eilenberg-Maclane space $K(\pi, n)$ is a CW-complex such that $\pi_q(K(\pi,n)) = 0$ if $q \neq n$ and $\pi_n(K(\pi,n)) = \pi$.

Remark 7.37. The homotopy type of $K(\pi, n)$ is totally determined by π and n. A concrete cellular construction of $K(\pi, n)$ is taking a wedge sum of n-spheres as generators of π, then gluing boundaries of $n+1$-disks on the representatives of the relations, then gluing higher cells to kill all the homotopy groups $\pi_q(X)$ for $q > n$.

Actually, up to homotopy equivalence, it is independent of the choice of concrete construction.

Definition 7.38. We take the concrete cellular construction of $K(\pi, n)$ mentioned in Remark 7.37 as the model.

Let $c \in C^n(K(\pi,n);\pi)$ be the cochain that assigns each n-cell to the corresponding elements in π. we claim that c is cocycle and we will prove the claim in Lemma. Then the cohomology class F_π represented by c is called **the fundamental class**.

Lemma 7.39. *c is a cocycle.*

Proof. Note that the $n+1$-cells of $K(\pi, n)$ corresponding to the relations of the generators. Suppose an $n+1$-cell corresponds to the relation $\sum_i k_i g_i = 0$ for $k_i \in \mathbb{Z}$ and g_i is a generator and e_i is the cell corresponding to g_i, then according to the coboundary formula 26, we have

$$\delta c(\sigma) = \sum_i [\partial\sigma : e_i] c(e_i) = \sum_i k_i g_i = 0$$

□

Proposition 7.8. $F_\pi = [d(\mathrm{const}, \mathrm{id})]$

Proof. For any n-cell $\sigma \in K(\pi, n)$, we have

$$d(\mathrm{const}, \mathrm{id}) \cdot \sigma = [d(\mathrm{const}, \mathrm{id})_\sigma]$$

where $\sigma \cong S^n = D^n_+ \cup S^{n-1} \cup D^n_-$ and $d(\mathrm{const}, \mathrm{id})_\sigma(D^n_+ \cup S^{n-1})$ is a constant map and actually $d(\mathrm{const}, \mathrm{id})_\sigma$ is homotopic to the inclusion map of σ, so $[d(\mathrm{const}, \mathrm{id})_\sigma]$ is the element in π corresponding to σ. □

Theorem 7.40 (Hopf-Whitney)**.** *Let K be a complex of dimension n, let Y be an $(n-1)$-connected space. Then*

$$\begin{array}{rccc} k: & [X, K(\pi, n)] & \longrightarrow & H^n(X; \pi) \\ & [f] & \longmapsto & f^*(F_\pi) \end{array} \tag{28}$$

Proof. First, we show k is surjective. Let u be a cohomology class in $H^n(X; \pi)$ and let c be a representative cocycle of u. By Proposition 7.7, there exists a map $f : X^n \to K(\pi, n)$ such that $f(X^{n-1})$ is a 0-cell of $K(\pi, n)$ and $d(\mathrm{const}, f) = c$. Then we may extend f to X, since the obstructions in $\pi_q(K(\pi, n)) = 0$ are trivial for all $q > n$. Note that $f^\# : C^n(K(\pi, n); \pi) \to C^n(X; \pi)$ maps $d(\mathrm{const}, \mathrm{id}) \mapsto d(\mathrm{const}, f) = c$, which is what we need.

Second, we show k is injective. Suppose f, g are two maps from X to $K(\pi, n)$ with $f^*(F_\pi) = g^*(F_\pi)$. By cellular approximation, we may assume f, g are cellular. In particular, $f|X^{n-1} = g|_{X^{n-1}} = \mathrm{const}$. Note that $d(f, g) = d(\mathrm{const}, g) - d(\mathrm{const}, f) = 0$, by Theorem 7.35, $f \simeq g$. □

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