# Obstructions to Realizing Homology Classes by Manifolds

TONGTONG LIANG

September 19, 2022

## ABSTRACT

This is a survey on Thom's solution to the Steenrod problem that is asking whether each homology class of a finite complex can be realized as a manifold. In particular, we clarify some vague arguments and calculations in Thom's paper. Following Thom's method, We first show how the problem is translated into a homotopy lifting problem by Thom's construction, then we calculate the obstructions of the corresponding lifting problems in terms of Steenrod operations. This survey aims to understand this method essentially, which is expected to enlighten us to think about how to generalize it to algebraic-geometric setting.

## CONTENTS



**Incomplete Draft**

*Department of Mathematics, Southern University of Science and Technology, Shenzhen, China,*
*12132882@mail.sustech.edu.cn*

# 1 INTRODUCTION

# 2 REALIZING HOMOLOGY CLASSES BY UNORIENTED MANIFOLDS

**Definition 2.1** A smooth manifold $M^n$ has a G-structure if its tangent bundle $TM$ is a G-bundle (or say $TM$ be reduced to a G-bundle).

**Definition 2.2** (Orientation) Given a commutative ring $R$, a $R$-fundamental class of a manifold $M^n$ is a homology class $\sigma \in H_n(M^n; R)$ such that for any $p \in M^n$, the image of $\sigma$ is a generate of $H_n(M^n, M^n - p; R) \cong R$. If $M^n$ admits a $R$-fundamental class, then we say $M^n$ is $R$-orientable. If we say $M^n$ is orientable without mentioning the ring, we mean $M^n$ is $\mathbb{Z}$-orientable by default.

In this section, we just discuss the unoriented case. Note that every manifold is $\mathbb{F}_2$-orientable, so in this section, the (co)homology groups are all set to be $\mathbb{F}_2$-coefficient and $H_*(-) := H_*(-; \mathbb{F}_2)$ (similar for cohomology). Let $[M^n] \in H_n(M^n)$ denote the fundamental class of $M^n$ if we denote specify a fundamental calss.

**Definition 2.3** (Homological realization) We say $u \in H_k(M^n)$ can be realized by a G-submanifold, if there exists a G-submanifold $W^k$ of $M^n$ with the natural inclusion $i\colon W^k \to M^n$ such that $u = i_*[W^k]$.

**Conjecture 2.4** **Is any $u \in H_k(M^n)$ is $O(k)$-realizable for any positive integer $k$ and manifold $M^n$?**

To study this conjecture, we now reformulate the question in cohomological version. This observation is given by Thom.

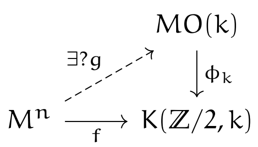
**Definition 2.5** (Cohomological realization) We say a cohomology class $u \in H^k(X)$ is G-realizable if there exists a map $f\colon X \to MG$ such that $f^*(\Phi_G) = u$, where $MG$ is the Thom space of $G$ and $\Phi_G$ is the universal Thom class in $H^k(MG)$.

By Pontrjagin-Thom construction, we will see that these two notion of realziation coincide via Poincare duality.

**Theorem 2.6** (Thom) Suppose $M^n$ is a manifold, then $u \in H^k(M^n)$ is G-realizable if and only if its dual $z \in H_{n-k}(M^n)$ is G-realizable.

*Proof.* See [Tho54]. □

By using this theorem, Conjecture 2.4 is turned in a homotopical lifting problem: let $f\colon M^n \to K(\mathbb{Z}/2, k)$ be the map corresponding to the cohomology class $u$ and $\phi$ be a map induced by Thom class, then we will have the following lifting problem:




(1)

The lifting problem is asking whether there exists a continuous map $g\colon M^n \to MO(k)$ such that $\phi \circ g \simeq f$. Note that the existence of such $g$ is equivalent to a positive answer of Conjecture 2.4, because $g^*\phi^*\iota_k = g^*(\Phi_{O(k)}) = f^*\iota_k$, where $\iota_k$ is the fundamental class of the Eilenberg-Mac lane space, if and only if $\phi \circ g \simeq f$, according to the Brown's representability of ordinary cohomology. In summary, the Conjecture 2.4 is equivalent to ask whether the lifting problem 1 admits an answer up to homotopy. Furthermore, the obstruction of the lifting problem is the obstruction to the realization problem. To disclose the obstruction, it is inevitable to study the homotopy type of $MO(k)$.

## 2.1 Computing the homotopy type of $MO(k)$

The homotopy groups is almost impossible to compute directly. Fortunately, we can use Whitehead theorem reduce the computation on cohomology groups for simply connected spaces.

**Theorem 2.7** (Whitehead) Suppose $X, Y$ are two simply connected CW-complexes and let $f : X \to Y$ be a cellular map such that for any prime $p$, $f^* : H^r(Y;\mathbb{F}_p) \to H^r(X;\mathbb{F}_p)$ is an isomorphism for $r < k$ and a monomorphism for $r = k$. Then there exists a cellular map $g : Y \to X$ such that $f, g$ are homotopy equivalence on $(k-1)$-skeletons. (There exists homotopies from $f|_{X_{k-1}} \circ g|_{Y_{k-1}}$ and $g|_{Y_{k-1}} \circ f|_{X_{k-1}}$ to the identities)

Note that $MO(k)$ and $K(\mathbb{Z}/2, k)$ are simply connected for $k > 1$.

**Remark 2.8** Since if $p$ is an odd prime, $H^*(\mathbb{Z}/2, k;\mathbb{F}_p)$ and $H^*(BO(k);\mathbb{F}_p)$ are trivial and $H^*(BO(k); R) \cong H^{*+k}(MO(k); R)$ via Thom isomorphism, thus we just need to consider the mod 2 cohomology ring of $MO(k)$ and $K(\mathbb{Z}/2, k)$.

### 2.1.1 *The cohomology of* $K(\mathbb{Z}/2, k)$

We simply denote $H^*(K(G, n); A)$ by $H^*(G, h; A)$.

**Theorem 2.9** (Serre-Cartan) $H^*(\mathbb{Z}/2, k;\mathbb{F}_2)$ is generated by iterated Steenrod squares on the fundamental class $\iota \in H^k(\mathbb{Z}/2, k;\mathbb{F}_2)$. More specifically, $H^{k+h}(\mathbb{Z}/2, k;\mathbb{F}_2)$ is generated by

$$\{Sq^{i_1} \cdots Sq^{i_r}(\iota) \mid \sum_{m=1}^{r} i_m = h\}$$

An **admissible sequence** $I$ is an ordered sequence with finitely many positive integers $\{i_1, \dots, i_r\}$ such that $i_1 \geqslant 2i_2, \dots, i_{r-1} \geqslant 2i_r$. We denote $Sq^I = Sq^{i_1} \dots Sq^{i_r}$. The **total degree** of the sequence $I$ is defined to be $\sum_m i_m$. The **length** of $I$ is the number of non-zero elements in $I$.

**Proposition 2.10** The admissible iterated Steenrod squares of the fundamental class form a basis of $H^*(\mathbb{Z}, k)$.

**Proposition 2.11** The number of decomposition (ignoring the order) of $h$ in summand of type $2^m - 1$, which is called **dyadic decomposition of** $h$, is equal to the number of admissble sequence of total degree $h$.

*Proof.* Given an admissible sequence $\{i_1, \dots, i_r\}$ and we let $j_n = i_n - i_{n-1}$ for $n < k$ and $j_r = i_r$. Note that all these $j_n$ are non-negative and we have

$$i_n = \sum_{k=n}^{r} 2^{k-n} j_k \tag{2}$$

then

$$h = \sum_k i_k = \sum_k (2^k - 1) j_k \tag{3}$$

which is a decomposition of $h$ of type $2^k - 1$. Conversely, given such a decomposition of $h$, we may write it into the form of $j_i$ as formula (3) uniquely, then define $i_k$ as formula (2). □

### 2.1.2 *The cohomology ring of* $MO(k)$

Note that $BO(k)$ has a manifold model $G_k$ called $k$-**real Grassmannian manifold** and the cohomology rings is

$$H^*(G_k) = \mathbb{F}_2[W_1, \dots, W_k]$$

where $W_i$ is the i-th universal Stiefel-Whitney class of degree i. The details are in [MS74].

Let J be the ideal of $H^*(G_k)$ generated by $W_k$, then $H^*(MO(k)) \cong J$ via the Thom isomorphism. Now the cohomology ring structure of $MO(k)$ is clear, but it is not enough to show solve the lifting problem 1. We still need to consider the action of Steenrod algebra $\mathcal{A}_2$ on $H^*(MO(k))$, more specifically, we need to consider the submodule generated by the Thom class in $H^*(MO(k))$.

**Lemma 2.12** (Wu formula) For any integer r, we have

$$Sq^r W_i = \sum_t \binom{r-i+t-1}{t} W_{r-t} W_{i+t}$$

To study the action of the iterated Steenrod squares on $W_k$, we define a lexicographic order (R) on monomials in variables with $W_1, \dots, W_k$ by setting $W_m < W_n$ if $m < n$. For example,

$$W_5 < W_5 W_1 W_1 < W_5 W_1 W_4 < W_5 W_3 < W_6$$

Then we have the following lemma to show the action more precisely:

**Lemma 2.13** For any admissible sequence $I = \{i_1, \dots, i_r\}$ of total degree h, there exists $Q_I \in H^h(G_k)$ such that $Sq^I W_k = W_k \cdot Q_I$, where

$$Q_I = W_{i_1} W_{i_2} \dots W_{i_r} + \text{strictly lower terms with respect to (R)}$$

.

*Proof.* The idea of the proof is to iterate Wu's formula and we argue it by induction on the length of I. Let $I = \{i_1, \dots, i_r\}$. First, if $r = 1$, then $Sq^i W_k = W_k W_i$, which is the desired result. Suppose the assertion is true for $r < n-1$. Then

$$\begin{aligned} Sq^I W_k &= Sq^{i_1}(Sq^{i_2} \dots Sq^{i_r} W_k) = Sq^{i_1}(W_k P) \\ &= \sum_{m=0}^{i_1} Sq^m(P) \cdot Sq^{i-m} W_k \\ &= \sum_{m=0}^{i_1} Sq^m(P) W_{i-m} W_k = W_k \Big( \sum_{m=0}^{i_1} Sq^m(P) W_{i-m} \Big) \end{aligned}$$

According to the inductive hypothesis, we have $P = W_{i_2} \dots W_{i_r} +$ strictly lower terms. Then we may let $Q_I = \sum_{m=0}^{i_1} Sq^m(P) W_{i_1 - m}$ and when $m = 0$, we have $Q_I = W_{i_1} W_{i_2} \dots W_{i_r} + \sum_{m=1}^{i_1} Sq^m(P)$. Note that $Sq^m(P)$ only contains those classes $W_i$ for which $i < 2i_2 \leqslant i_1$ according to Wu's formula. □

**Corollary 2.14** Any linear combination of iterated Steenrod squares $Sq^I$ of total degree $h \leqslant k$, which vanishes on $W_k$, is 0.

*Proof.* Since the admissible iterated Steenrod squares form a basis of $\mathcal{A}_2$, we may reduce the case to the admissible cases. According to Lemma 2.13, we have that $Sq^I W_k = W_k \cdot Q_I$. Note that the monomials with distinct order with respect to (R) are linearly independent. Thus we just need to check the linear dependence of the leading term of $Q_I$: $W_{i_1} \dots W_{i_r}$, then we have the result immediately. □

By the splitting principle, we may take $H^*(BO(k))$ such a subalgebra of $H^*(\mathbb{RP}^\infty)^k = \mathbb{F}[t_1, \dots, t_k]$, where $t_i$ is a the first Stiefel-Whitney class of the universal line bundle at the i th factor and $W_i$ corresponds to the i th elementary symmetric polynomial. Thus we have

**Lemma 2.15** $Sq^I(t_1 \dots t_k)$ are linearly independent for all admissible sequences of total degree $h \leqslant k$.

We observe that $H^h(BO(k))$ has a basis consists

$$\{\sum(t_1)^{a_1}\dots(t_k)^{a_k} \mid \sum a_i = h\}$$

where the notation $\sum(t_1)^{a_1}\dots(t_k)^{a_k}$ means the homogeneous symmetric polynomial containing the monomial summand $(t_1)^{a_1}\dots(t_k)^{a_k}$ and is totally determined by the $k$-partition of $h$ (the ways how we decompose $h$ into a sum of $k$ non-negative integers). Therefore, for any decomposition $\omega = \{a_1, \dots, a_k\}$, **let $S_\omega$ to denote the essential system of permutations on** $\omega$. For example for $\omega = \{1, 1, 2\}$,

$$S_\omega = \{(1, 1, 2), (1, 2, 1), (2, 1, 1)\}$$

Notice that the tuple $(-, \dots, -)$ is ordered and $\{-, \dots, -\}$ is not. Then the sum $\{\sum(t_1)^{a_1}\dots(t_k)^{a_k}$ is indexed by $S_\omega$ actually.

Recall that we identify $H^*(MO(k))$ with the ideal $J$ in $\mathbb{F}_2[t_1, \dots, t_k]$ generated by $t_1t_2\dots t_k$, thus a basis for $H^{h+k}(MO(k))$ has a basis

$$\sum(t_1)^{\alpha_1} + 1)(t_2)^{\alpha_2+1}\dots(t_r)^{\alpha_r+1}t_{r+1}\dots t_k$$

and the dimension of $H^{h+k}(MO(k))$ is the number of partitions of $h$. The following context is to study the relation of these elements with respect to Steenrod squares. To show the relation precise, we need the following convention:

**Definition 2.16** Suppose $P \in \mathbb{F}_2[t_1, \dots, t_k]$, then we say $t_i$ is **dyadic** for $P$ if the exponent of $t_i$ is either $0$ or $2^m$ for non-negative integer $m$.

**Example 2.17** For $P(t_1, t_2, t_3) = t_3^3 + t_1^2t_2 + t_1t_3 + t_1$, $t_1$ and $t_2$ are dyadic while $t_3$ is non-dyadic.

**Lemma 2.18** If $t_i$ is dyadic for $P$, then $t_i$ is dyadic for $Sq^jP$.

*Proof.* Note that $Sq^a(t_i)^m = \binom{m}{a}t_n^{m+a}$. If $m \neq 0$ and $m = 2^r$, then $\binom{m}{a} \equiv 0 \mod 2$ except to $a = 0$ or $a = m$. Therefore only when $m = 2^r$ and $a = 2^r$, $t_n$ can survive after the operation. For general case, we may use Cartan's formula to do induction. □

**Definition 2.19** Suppose $P = t_1^{a_1}\dots t_r^{a_r}$ is a monomial, the **non-dyadic factor** of $P$ is the monomial consisting of all non-dyadic variables. Further, we may write $P = ND$, where $N$ consists of non dyadic variables and $D$ consists of dyadic variables. We use $u(P)$ to denote the number of non-dyadic variables and $\nu(P) = \deg(N)$. An order $(N)$ among the monomials in $\mathbb{F}_2[t_1, \dots, t_k]$ is defined to be: $P_1 > P_2$ if $u(P_1) > u(P_2)$ or $u(P_1) = u(P_2), \nu(P_1) < \nu(P_2)$.

We denote

$$X_\omega^h = \sum(t_1)^{a_1+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_k$$

for any $h \leqslant k$ and **non-dyadic decomposition** $\omega = \{a_1, \dots, a_r\}$ **of** $h$ **i.e** $a_i \neq 2^m - 1$ for each $i$.

**Lemma 2.20** The (N)-leading term of $Sq^IX_\omega^h$ is of the form

$$\sum_{S_\omega}(t_1)^{a_1+1}\dots(t_r)^{a_r+1}Sq^I(t_{r+1}\dots t_k)$$

*Proof.* Note that $t_{r+1}, \dots, t_k$ are dyadic in $(t_1)^{a_1+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_k$, then by Lemma 2.18, they are still dyadic in $Sq^I(t_1)^{a_1+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_k$. Hence

$$u(Sq^I(t_1^{a_1+1}\dots(t_r)^{a_r+1}t_{r+1}\dots t_k)) \leqslant r \tag{4}$$

Iterating Cartan's formula, the monomials $P$ in the polynomials of the equality 4 with $u(P) = r$ appear in the polynomial

$$t_1^{a_1+1} \dots t_r^{a_r+1} Sq^I(t_{r+1} \dots t_k) \quad \text{(Type 1)}$$

$$Sq^{I'}(t_1^{a_1+1} \dots t_r^{a_r+1}) Sq^{I''}(t_{r+1} \dots t_k) \quad \text{(Type 2)}$$

Note that the index $\nu$ of Type 1 polynomial is $r+h$ and the index $\nu$ of Type 2 polynomial is strictly less than $r+h$ because the Steenrod squares lift the their degree of dyadic factors (note that $|I| > |I''|$). Therefore, by taking the sum indexed by all the essential permutations, we have the result. □

**Lemma 2.21** For any $m \leqslant k$,

$$\{X^m_{\omega_m}, Sq^I X^{m-1}_{\omega_{m-1}}, Sq^{I_h} X^h_{\omega_h}, \dots, Sq^I W_k\} \quad \text{(B)}$$

where $I_h$ runs out the admissible sequences of total $m-h$ and $\omega_h$ runs out the non-dyadic decomposition of $h$.

**Observation**: No term can be expressed as a linear combination of strictly lower terms with respect to the order (N). Thus we just need to check the linear independence for $Sq^I X^h_{\omega_h}$, whose leading terms have the same order with respect to (N).

*Proof.* According to the observation, we reduce the case to fixed $r$, $h$ and a specified order of variables (because if a polynomial is non-zero, its sum of all permutations is non-zero). Suppose

$$\sum_{|I_\lambda|=h} c_\lambda (t_1)^{a_1+1} \dots (t_r)^{a_r+1} Sq^{I_\lambda}(t_{r+1} \dots t_k) = 0$$

then we have

$$(t_1)^{a_1+1} \dots (t_r)^{a_r+1} \sum_{|I_\lambda|=h} c_\lambda Sq^{I_\lambda}(t_{r+1} \dots t_k) = 0$$

By Lemma 2.15, we conclude that $c_\lambda = 0$ for each $\lambda$. □

**Corollary 2.22** The set (B) Lemma 2.1.2 forms a basis of $H^{m+k}(MO(k))$.

*Proof.* Recall that the $H^{m+k}(MO(k))$ is isomorphic to $J^{m+k}$. Note that

$$\dim J^{m+k} = p(m),$$

the number of decomposition of $m$. The number of elements in (B) is

$$\sum_{h \leqslant m} d(h) c(m-h)$$

where $d(h)$ is the number of non-dyadic decomposition of $h$ and $c(m-h)$ is the number of dyadic decomposition of $m-h$, because $d(h)$ corresponds to the enumeration of $\omega_h$ and $c(m-h)$ corresponds to the number of admissible sequences of total degree $m-h$ by Proposition 2.11. By directly counting, we have

$$p(m) = \sum_{h \leqslant m} d(h) c(m-h)$$

and by Proposition 2.1.2, we complete the proof. □

### 2.1.3 *The homotopy type of* $MO(k)$

Recall that $[X, K(A, n)] \cong H^n(X; A)$ via Hopf-Whitney's theorem. Then we let

$$F_\omega \colon MO(k) \to K(\mathbb{F}_2, k+h)$$

be a map that represents the homotopy class corresponding to the cohomology class $X^h_\omega \in H^{k+h}(MO(k))$. Then we have

$$F \colon MO(k) \to Y := K(\mathbb{F}_2, k) \times K(\mathbb{F}_2, k+2) \times \cdots \times K(\mathbb{F}_2, 2k)^{d(k)}$$

by taking the product of $F_\omega$ for any $h \leqslant k$ and non-dyadic decomposition $\omega$ of $h$. Since (B) is a basis of $H^{k+h}(MO(k))$, $F^*$ is an isomorphism from $H^{k+m}(Y; \mathbb{F}_2)$ to $H^{k+m}(MO(k); \mathbb{F}_2)$ for $m \leqslant k$. Then by Whitehead's theorem and Remark 2.8, we conclude that $F \colon MO(k) \to Y$ is a $2k$-equivalence. Then we let $g \colon Y \to MO(k)$ be the $2k$-equivalence inverse of $F$ and if we restrict $g$ to the first factor, then we have

$$g_k \colon K(\mathbb{F}_2, k) \to MO(k)$$

such that $g_k^*(\Phi_k) = \iota$ (recall that the $W_k$ corresponds to the Thom class $\Phi_k$ via the Thom isomorphism and $W_k$ corresponds to $\iota_k$ according to the definition of $F$ and $g$). In this way, the lifting problem 1 admits a solution if $n \leqslant 2k$. In other words, if $u \in H^k(A; \mathbb{F}_2)$ and $\dim A \leqslant 2k$, then $u$ is $O(k)$-realizable. For a manifold $M^n$, the duality theorem provides us with

$$H^k(M^n) \cong H^{n-k}(M^n)$$

and by Theorem 2.6, if

$$n \leqslant 2n - 2k$$

then any homology class in $w \in H_k(M^n)$ can be realized by a submanifold. Thus we have the following theorem

**Theorem 2.23** Given a manifold $M^n$, for any $k \leqslant n/2$ and any $u \in H^k(M^n; \mathbb{F}_2)$ can be realized as a submanifold with inclusion.

## 2.2 On the unoriented Steenrod-Thom theorem

In this subsection, we use the result in previous section to prove the positive answer the unoriented Steenrod conjecture:

**Conjecture 2.24** (Steenrod) For any finite simplicity complex $K$(polyhedron) and any homology class $u \in H_r(K; \mathbb{F}_2)$, is there a manifold $M^r$ with a continuous map $f \colon M^n \to K$ such that $u = f_*[M^n]$?

For a finite polyhedron $K$ of dimension $m$, $K$ can be embedded in $\mathbb{R}^n$ for $n \geqslant 2m+1$ linearly. Then we have a regular open neighbourhood $U$ of $K$ in $\mathbb{R}^n$ such that $K$ is a retract of $U$ and the boundary of $U$ is an $n-1$-manifold (details in [Tho54]).Then let $M^n$ be a closed manifold obtained by gluing two copies of $U$ along the boundary

$$M^n = U \sqcup U / \partial U$$

Let $j : U \to M^n$ be an inclusion and let $q : M^n \to U$ be the folding map by identifying the same points of two copies of $U$, then $q \circ j = \mathrm{id}_U$ and thus $U$ is a retract of $M^n$. Furthermore, $K$ is a retract of $M^n$. We let $i \colon K \hookrightarrow M^n$ be the inclusion and $r \colon M^n \to U$ be the retraction with respect to $i$.

**Theorem 2.25** For any $u \in H_r(K)$ for $r \leqslant m = \dim K$, there exists a manifold $V^r$ with a continuous map $f \colon V^r \to K$ such that $f_*[V^r] = u$.

*Proof.* Since $i \colon U \to M^n$ is a retract, then we may embed $H_r(U)$ into $H_r(M^n)$. Note that $r < m < n/2$, then $i_* u$ can be realized by a submanifold $V^r$ of $M^n$ by Theorem

2.23. Then let $f := r \circ r|_{V^r} : V^r \to U$ is the desired map such that $f_*[V^r] = u$ because of the community of the following diagram:

$$\begin{array}{ccc} K & \hookrightarrow & U \\ {\scriptstyle f}\uparrow & \overset{i}{\searrow} \; \underset{r}{\nwarrow} & \downarrow{\scriptstyle j} \\ V^r & \hookrightarrow & M^n \end{array}$$

□

# 3 REALIZING HOMOLOGY CLASSES BY ORIENTED MANIFOLDS

In this section, we discuss the realization problem for integral homology classes by oriented submanifolds. Recall Theorem 2.6, the realization problem is actually a homotopy lifting problem

$$\begin{array}{ccc} & & MSO(k) \\ & {\scriptstyle \exists ? g} \nearrow & \downarrow{\scriptstyle \phi_k} \\ M^n & \xrightarrow[f]{} & K(\mathbb{Z}, k) \end{array} \tag{5}$$

where $M^n$ is an oriented manifold.

In last section, we solve the lifting problem by compute the stable homotopy type of $MO(k)$. We see that the $2k$-homotopy type of $MO(k)$ is a free product of Eilenberg Mac Lance spaces. However, the homotopy type of $MSO(k)$ is much more complicated. The first problem is that $MSO(k)$ is not a free product of Eilenberg Mac Lane spaces and the second problem is that the homotopy groups of $MSO(k)$ is not $p$-local for some prime $p$. To proceed the computation, we need to introduce Eilenberg Mac Lane $k$-invariants and $\mathbb{Z}/p$-Steenrod operations for all primes $p$.

## 3.1 Twisted product between Eilenberg-MacLane spaces

In last section, we compute that the stable homotopy type of $MO(k)$ is a direct product of Eilenberg Mac Lane space. However, in general, given a simply connected space $X$, it is not homotopy equivalent to the product $K = \prod_i K(\pi_i(X), i)$, even in stable case. The question is: how to measure the distance between $X$ and $K$? Let's begin with some baby cases.

Suppose $X$ is a space with two non-trivial homotopy groups with $\pi_k(X) = A$ and $\pi_{k+n}(X) = B$ for some $1 < k$ and $n > 0$. Obviously, $K(A, k) \times K(B, k+n)$ has the same homotopy groups as $X$. Recall the fibration

$$\Omega K(B, k+n+1) \cong K(B, k+n) \hookrightarrow PK(B, k+n+1) \to K(B, k+n+1)$$

where $PK(B, k+n)$ is the space of paths in $K(B, k+n)$ with compact-open topology. Then for any $h\colon K(A, k) \to K(B, k+n+1)$, we have the following pull-back diagram

$$\begin{array}{ccc} Y^f & \xrightarrow{g} & PK(B, k+n+1) \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ K(A, k) & \xrightarrow[h]{} & K(B, k+n+1) \end{array} \tag{6}$$

where $Y^h$ is a simple fiber bundle on $K(A, k)$ with fiber $K(B, k+n)$. Then using the long exact sequence associated to the the fibration, we conclude that $Y^h$ has the same homotopy groups as $K(A, k) \times K(B, k+n)$. Furthermore, if $h \sim h'$, then $Y^h \simeq Y^{h'}$, since we may view $K(B, k+n+1)$ as a classifying space of principal

$K(B, k+n)$-bundle. The following proposition indicates that such construction $Y^h$ runs out all the homotopy type with such homotopy groups.

**Proposition 3.1** There is bijection between the set of homotopy types with $\pi_k = A$ and $\pi_{k+n} = B$ and $H^{k+n+1}(A, k; B)$

*Proof.* According to the classification theorem of principal bundles, we just need to show there is a bijection between the set of isomorphic classes of principal $K(B, n)$-bundles on $K(A, k)$ and the set of homotopy types with $\pi_k = A$ and $\pi_{k+n} = B$.

First, we show that every such homotopy type has a representative of such principal bundle. Given such a $X$ with $\pi_k(X) = A$ and $\pi_{k+n}(X) = B$. By attaching higher cells to kill elements in $\pi_{k+n}(X)$, we have a map $q : X \to K(A, k)$ that induces isomorphism between the $k$-th homotopy groups. Then we may replace $q$ by a fibration as

$$X \xrightarrow{e} N(q) \xrightarrow{p} K(A, k)$$

where $e$ is a homotopy equivalence and $p$ is a fibration we need and we may view $N(q)$ as a fiber bundle on $K(A, k)$ with fiber $K(B, k+n)$ (the long exact sequence of homotopy groups tells us that the fiber of $p$ is homotopy equivalent to $K(B, k+n)$).

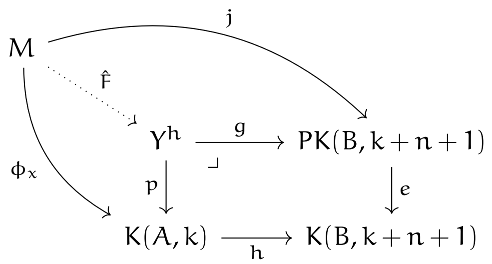
Conversely, given such a principal bundle $E$, we just take its total space. □

**Remark 3.2** From this perspective, we see how cohomology operations determine homotopy types in some special case. More specifically, if we compare the following two fibration $Y^h \to K(A, k)$ as a twisted projection $K(A, k) \times K(B, k+n) \to K(A, k)$, we may view $Y^h$ as a product between $K(A, k)$ and $K(B, k+n)$ twisted by a cohomology operation $\phi_h : H^k(-; A) \to H^{k+n}(-; B)$ represented by $h : K(A, k) \to K(B, k+n+1)$. In other words, this twist can be viewed as an obstruction for an extension problem.

The following two lemmas will be used in later subsections.

**Lemma 3.3** Suppose $M$ is a CW-complex and $\phi_x : M \to K(A, k)$ is a cellular, then $\phi_x$ admits a lifting along $Y^h \to K(A, k)$ if and only if $\phi_h(x) = 0$, where $x = \phi_x^* \iota \in H^k(M; A)$.

*Proof.* Suppose $\phi_h(x) = 0$ i.e $h \circ \phi_x$ is null-homotopic, then we can lift $h \circ \phi_x$ to $j : M \to PK(B, k+n+1)$ along the natural projection. Then the universal property of the following pull-back diagram provides us with $\hat{F} : M \to Y^h$.




Note that the map $\hat{F} : M \to Y^h$ is indeed a lift of $\phi_x$, since the diagram is commute.

Conversely, if we can lift $\phi_x$ to the whole $Y^h$, the map $\phi_x$ that represents $x$ is null-homotopic after composition with $h$ according to the construction of $Y^h$, which means that $\phi_h(x) = 0$. □

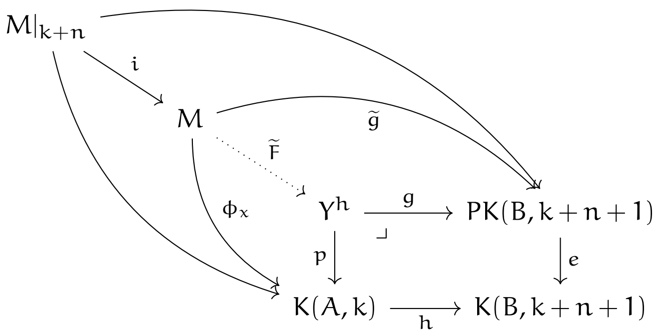
**Lemma 3.4** Let $F : M \to Y^h$ be a map defined on $(k+n)$-skeleton, and let $x \in H^k(M|_{k+n}; A) \cong H^k(M; A)$ be the cohomology class represented by $p \circ F$. Then we can extend $F$ to the whole $M$ if and only if $\phi_h(x) = 0$.

*Proof.* According to obstruction theory, we can extend $p \circ F : M|_{k+n} \to K(A, k)$ to $\phi_x : M \to K(A, k)$, because $\pi_m(A, k) = 0$ for $m \geqslant k+n$. Similarly, we can extend $g \circ F : M|_{k+n} \to PK(B, k+n+1)$ to $\widetilde{g} : M \to PK(B, k+n+1)$ such that $e \circ \widetilde{g} =$

$h \circ \phi_x$ by using the homotopy extension property with respect to the cofibration $i \colon M|_{k+n} \to M$.

$$\begin{array}{ccc} M|_{k+n} & \xrightarrow{g \circ F} & PK(B, k+n+1) \\ {\scriptstyle i}\downarrow & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle e} \\ M & \xrightarrow[h \circ \phi_x]{} & K(B, k+n+1) \end{array}$$

Then using the universal property of the pull-back, we have $\tilde{F} \colon M \to Y^h$ to extend F




Note that this is indeed an extension of F, due to the uniqueness given by the universal property. □

### 3.2 Auxiliary construction: Silber's polyhedron

Differently from the case of $MO(k)$ where we just need to consider $\mathbb{F}_2$ coefficient cohomology and Steenrod squares, we need to consider cohomology operations with coefficient $\mathbb{Z}$ and $\mathbb{F}_p$ for odd prime p in the case of $MSO(k)$. Specifically, we need Bockstein long exact sequence and Bockstein operations to help us derive integral power operations from mod p Steenrod power operations.

Suppose we have a short exact sequence of abelian groups:

$$0 \longrightarrow G \longrightarrow H \longrightarrow K \longrightarrow 0 \tag{7}$$

then we have an induced short exact sequence of chain complexes

$$0 \longrightarrow C^*(X; G) \longrightarrow C^*(X; H) \longrightarrow C^*(X; K) \longrightarrow 0$$

then by the Snake lemma, we have a long exact sequence

$$\cdots \longrightarrow H^n(X; G) \longrightarrow H^n(X; H) \longrightarrow H^n(X; K) \longrightarrow H^{n+1}(X; G) \longrightarrow \cdots$$

The connected morphism $H^n(X; K) \to H^{n+1}(X; G)$ is so called **Bockstein homomorphism** associated to the short sequence 7.

We let $\beta \colon H^n(X; \mathbb{F}_p) \to H^{n+1}(X; \mathbb{F}_p)$ be the Bockstein homomorphism associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

and we let $\tilde{\beta} \colon H^n(X; \mathbb{F}_p) \to H^n(X; \mathbb{Z})$ be the Bockstein operation associated to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

Let $q : H^n(X;\mathbb{Z}) \to H^n(X;\mathbb{Z}/p)$ be the morphism induced by the quotient map $\mathbb{Z} \to \mathbb{Z}/p$ and we have $\beta \circ q = \widetilde{\beta}$. The detail can be found in Allen Hatcher's book [Hat02].

Recall that given an odd prime $p$ and an integer $r > 0$, the Steenrod operation $St_p^{2r(p-1)+1} = \beta \circ \mathcal{P}_p^r$ where $\mathcal{P}_p^r$ is the reduced power operation. For such $St_p^{2r(p-1)+1}$, we can view it as an integral cohomology operation in the following way:

$$H^n(X;\mathbb{Z}) \xrightarrow{q} H^n(X;\mathbb{Z}/p) \xrightarrow{\mathcal{P}_p^r} H^{n+2r(p-1)}(X;\mathbb{Z}/p) \xrightarrow{\widetilde{\beta}} H^{n+2r(p-1)+1}(X;\mathbb{Z})$$

Here we abuse notation to let $St_p^{2r(p-1)+1}$ denote such integral power operation when it does not lead confusion.

Let $f\colon K(\mathbb{Z},k) \to K(\mathbb{Z},k+5)$ be the classifying map of $St_3^5$ as integral operation. Then the **Silber polyhedron** $K$ is a fiber bundle with base space $K(\mathbb{Z},k)$ and fiber $K(\mathbb{Z},k+4)$ twisted by $f$. In other words, $K = Y^h$ in the diagram 6 with suitable adjustment on symbols.

### *3.2.1 The cohomology ring of Silber's polyheron*

In the rest of the section, we let $\iota$ be the fundamental class of $K(\mathbb{Z},k)$ and $\nu$ be the fundamental class of $K(\mathbb{Z},k+4)$.

Let $F^3 : K(\mathbb{Z},k) \to K(\mathbb{Z},k)$ be the map classifying the cohomology operation operation $\cdot 3 : H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{Z})$, namely $(F^3)^*\iota = 3\iota$. Then we let $G$ be the pull-back bundle of $K \to K(\mathbb{Z},k)$:

$$\begin{array}{ccc} E & \xrightarrow{G} & K \\ \downarrow & \lrcorner & \downarrow \\ K(\mathbb{Z},k) & \xrightarrow{F^3} & K(\mathbb{Z},k) \end{array}$$

Note that $St_3^5\iota$ is an element of order 3, according to the Bockstein long exact sequence. Then we have

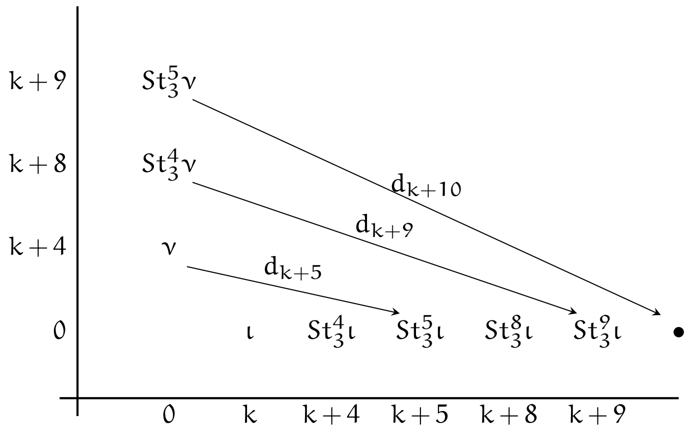
$$(F^3)^*St_3^5\iota = St_3^5(F^3\iota) = St_3^5(3\iota) = 0$$

which means that $E$ is a $K(\mathbb{Z},k+4)$-bundle with trivial twist, i.e. a trivial bundle $E = K(\mathbb{Z},k) \times K(\mathbb{Z},k+4)$.

With $\mathbb{F}_2$ coefficient, the induced morphism $(F^3)^*$ is an isomorphism, and then $G^*$ is an isomorphism on the $E^2$ pages of the corresponding Serre spectral sequences. Thus $G^*$ indcues an isomorphism between $H^*(K;\mathbb{F}_2)$ and $H^*(K(\mathbb{Z},k) \times K(\mathbb{Z},k+4);\mathbb{F}_2)$. Similarly, for $\mathbb{F}_p$ coefficient with $p > 3$, we still have the isomorphism

$$H^*(K;\mathbb{F}_p) \cong H^*(K(\mathbb{Z},k) \times K(\mathbb{Z},k+4);\mathbb{F}_p)$$

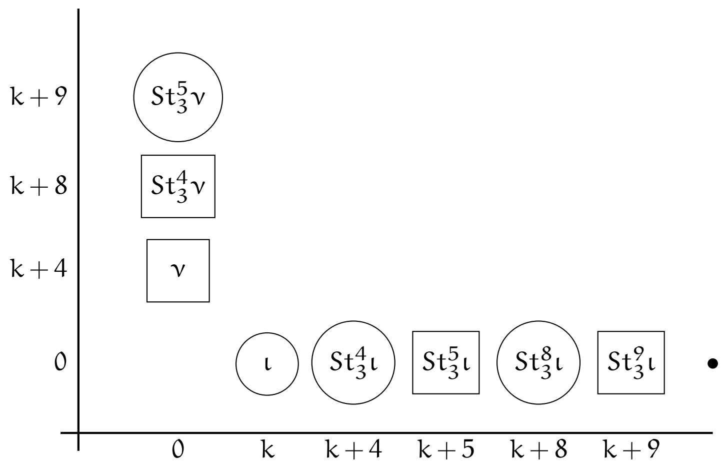
The subtle calculation is the case of $\mathbb{F}_3$ coefficient. The following diagram $E^2$-page with higher differentials of the fiber bundle $K$ in $\mathbb{F}_3$-cohomology:




The calculation of differentials are presented as follows:

- $d_{k+5}(\nu) = St_3^5\iota$ according to the construction of $K$;
- $d_{k+9}(St_3^4\nu) = St_3^4 d_{k+5}(\nu) = St_3^4 St_3^5\iota = St_3^9\iota$ because Steenrod operations commute with transgressions;
- $d_{k+10}(St_3^5\nu) = St_3^5 St_3^5\iota = 0$;

Passing to $E^\infty$-page, the circled elements in the following diagram survive to while the squared elements vanish.




Therefore, we have nontrivial cohomology groups of degree less than $k+10$ in $H^*(K;\mathbb{F}_3)$ as follows:

- $H^k(K;\mathbb{F}_3) = \mathbb{F}_3 \cdot \iota$;
- $H^{k+4}(K;\mathbb{F}_3) = \mathbb{F}_3 \cdot St_3^4\iota$;
- $H^{k+8}(K;\mathbb{F}_3) = \mathbb{F}_3 \cdot St_3^8\iota$;
- $H^{k+9}(K;\mathbb{F}_3) = \mathbb{F}_3 \cdot St_3^5\nu$;

In next subsection, we will use the cohomological properties of $K$ to compute the lower homotopy type of $MSO(k)$.

### 3.3 Computing the homotopy type of $MSO(k)$

In this subsection, our task is to use Theorem 2.7 and the Silber polyhedron $K$ to compute the lower homotopy type of $MSO(k)$.

Before we compute the homotopy type of $MSO(k)$, we need some information about cohomology groups of $MSO(k)$. By using Thom isomorphism, we just need to study the integral cohomology ring of $BSO(k)$.

**Theorem 3.5** $H^*(BSO(k);\mathbb{F}_2) = \mathbb{F}_2[W_2,\dots,W_k]$ where $W_i$ is the Stiefel-Whitney class of the universal bundle.

*Proof.* See Theorem 12.4 in [MS74]. □

**Theorem 3.6** If $R$ is an integral domain containing $1/2$, then the cohomology ring $H^*(BSO(2k+1);R)$ is a polynomial ring over $R$ generated by the Pontrjagin classes $P_1,\dots,P_m$ of the universal oriented bundle.

Similarly, $H^*(BSO(2k);R)$ is a polynomial ring over $R$ generated by the Pontrjagin classes $P_1,\dots,P_{m-1}$ and the Euler class $E$ of the universal oriented bundle.

*Proof.* See Theorem 15.9 in [MS74]. □

Brown gave the follow theorem to write down the integral cohomology ring explicitly in [Bro82]:

**Theorem 3.7** Let $\xi_k$ be the universal vector bundle over $BSO(k)$ and we just let $p_i = (-1)^i c_{2q}(\xi_k \otimes \mathbb{C})$ be the i-th Pontrjagin class of $\xi_k$; let $\widetilde{\beta}\colon H^*(BSO(k);\mathbb{Z}/2) \to H^{*+1}(BSO(k);\mathbb{Z})$ be the Bockstein homomorphism as we mentioned last subsection; let $w_i$ be the Stiefel-Whitney class of $\xi_k$ in $H^i(BSO(k);\mathbb{F}_2)$. Then we let

$$\mathcal{R}_k = \mathbb{Z}[p_1,\dots,p_{[(k-1)/2]}, X_k, \widetilde{\beta}(w_{2i_1}\cdot\ldots w_{2i_l} \mid 0 < i_1 < \cdots < i_l \leqslant [(k-1)/2])$$

and let $\mathcal{I}_k$ be an ideal in $\mathcal{R}_k$ generated by the following relations:

1. $2\widetilde{\beta}(w_{2i_1}\cdot\cdots\cdot w_{2i_l}) = 0$;
2. $X_k = \widetilde{\beta}(w_{2n})$ if $k = 2n+1$;
3. For $I = \{i_1,\dots,i_s\}$, we write $w(I) = w_{2i_1}\cdot\cdots\cdot w_{2i_s}$ and $p(I) = p_{i_1}\cdot\cdots\cdot p_{i_s}$, then the relation is

$$\widetilde{\beta}(w(2I))\cdot\widetilde{\beta}(w(2j)) = \sum_{i\in I}(\widetilde{\beta}w_{2i})\cdot p((I-\{i\})\cap J)\cdot\widetilde{\beta}(w(2((I-\{i\})\cup J-(I-\{i\})\cap J)))$$

Then

$$H^*(BSO(k);\mathbb{Z}) = \mathcal{R}_k/\mathcal{I}_k$$

Furthermore, if we let $q\colon H^*(BSO(k);\mathbb{Z}) \to H^*(BSO(k);\mathbb{Z}/p)$ be the quotient map, then $q(p_q) = W_{2q}^2$ in Theorem 3.5, if $p = 2$; $q(p_i) = P_i$ in $H^*(BSO(k);\mathbb{Z}/p)$ in Theorem 3.6 if $p$ is an odd prime.

By taking a suitable cellular decomposition, we may assume $K(\mathbb{Z},k)$ has only one $k$-cell and one $0$-cell that represents the fundamental class and $K(\mathbb{Z},k)$ has no $i$-cells for $0 < i < k$. With this setting, we have that $K$ and $K(\mathbb{Z},k)\times K(\mathbb{Z},k+4)$ have the same $(k+4)$-skeleton: it should be the $(k+4)$-skeleton of the base space $K(\mathbb{Z},k)$ plus the unique $(k+4)$-cell of $K(\mathbb{Z},k+4)$ over the only $0$-cell of $K(\mathbb{Z},k)$.

First, we have a cellular map

$$j\colon MSO(k) \to K(\mathbb{Z},k)\times K(\mathbb{Z},k+4)$$

induced by the integral Thom class $\Phi_{SO(k)}$ and $\Phi_{SO(k)}\cdot p_1 \in H^{k+4}(MSO(k);\mathbb{Z}) \cong \Phi_{SO(k)}\cdot H^4(BSO(k);\mathbb{Z})$. Since, $K(\mathbb{Z},k)\times K(\mathbb{Z},k+4)|_{k+4} = K|_{k+4}$, we have

$$j|_{k+4}\colon MSO(k)|_{k+4} \to K|_{k+4} \subset K$$

Note that $St_3^5\Phi_{SO(k)} = St_3^5(\Phi_{SO(k)}p_1) = 0$, because cohomology groups of $BSO(k)$ has no odd torsion (see Section 3 in [BS53], especially 3.1 and 3.5). Then by Lemma 3.4, we can extend $j|_{k+4}$ to $f\colon MSO(k) \to K$.

Recall that $H^*(BSO(k);\mathbb{F}_2) = \mathbb{F}_2[W_2,\dots,W_k]$ and $H^*(MSO(k);\mathbb{F}_2) = \Phi_{SO(k)}\cdot H^*(BSO(k)$ via Thom isomorphism. Therefore, we have $\Phi_{SO(k)}W_2W_3 \in H^{k+5}(MSO(k);\mathbb{F}_2)$ and we let

$$g\colon MSO(k) \to K(\mathbb{Z}/2, k+5)$$

to represent this cohomology class. Eventually, we have

$$F = f\times g\colon MSO(k) \to Y := K \times K(\mathbb{Z}/2, k+5)$$

**Proposition 3.8** $F\colon MSO(k) \to Y$ is a $(k+7)$-equivalence.

*Proof.* We will prove this proposition prime by prime according to Theorem 2.7.

We use the following convention in this proof:

- $\iota$: the image of the fundamental class of $K(\mathbb{Z}, k)$;
- $\nu$: the image of the fundamental class of $K(\mathbb{Z}, k+4)$;
- $\iota'$: the image of the fundamental class of $K(\mathbb{Z}/2, k+5)$;

**The $\mathbb{F}_2$-cohomology case**: The calculation in previous subsection shows that

$$H*(K;\mathbb{F}_2) \cong H^*(\mathbb{Z}, k;\mathbb{F}_2)\otimes H^*(\mathbb{Z}, k+4;\mathbb{F}_2)$$

then the image of $F^*$ can be written explicitly into the following table, according to the construction of $F$ and Theorem 3.7. Specifically, we have $F^*\iota = \Phi_{SO(k)}$, $F^*\nu = \Phi_{SO(k)}W_2^2$ and $F^*\iota' = \Phi_{SO(k)}W_2W_3$ according to the construction of F.

| Dimension | Generators in $H^*(Y;\mathbb{F}_2)$ | Image of $F^*$ in $H^*(MSO(k);\mathbb{F}_2)$ |
|---|---|---|
| k | $\iota$ | $\Phi_{SO(k)}$ |
| k+1 | 0 | 0 |
| k+2 | $Sq^2\iota$ | $\Phi_{SO(k)}W_2$ |
| k+3 | $Sq^3\iota$ | $\Phi_{SO(k)}W_3$ |
| k+4 | $Sq^4\iota$<br>$\nu$ | $\Phi_{SO(k)}W_4$<br>$\Phi_{SO(k)}(W_2)^2$ |
| k+5 | $Sq^5\iota$<br>$\iota'$ | $\Phi_{SO(k)}W_5$<br>$\Phi_{SO(k)}W_2W_3$ |
| k+6 | $Sq^6\iota$<br>$Sq^4Sq^2\iota$<br>$Sq^2\nu$<br>$Sq^1\iota'$ | $\Phi_{SO(k)}W_6$<br>$\Phi_{SO(k)}(W_2W_4+W_3^2+W_2^3)$<br>$\Phi_{SO(k)}(W_2^3+W_3^2)$<br>$=\Phi_{SO(k)}(W_3^2)$ |
| k+7 | $Sq^7\iota$<br>$Sq^5Sq^2\iota$<br>$Sq^3\nu$<br>$Sq^2\iota'$ | $\Phi_{SO(k)}W_7$<br>$\Phi_{SO(k)}(W_5W_2+W_4W_3+W_3W_2^2)$<br>$\Phi_{SO(k)}(W_3W_2^2)$<br>$\Phi_{SO(k)}W_2(W_5+W_3W_2)$ |
| k+8 | $Sq^8\iota$<br>$Sq^6Sq^2\iota$<br>$Sq^4\nu$<br>$Sq^3\iota'$<br>$Sq^2Sq^1\iota$ | $\Phi_{SO(k)}W_8$<br>$\Phi_{SO(k)}(W_6W_2+W_5W_3+W_4W_2^2))$<br>$\Phi_{SO(k)}(W_4W_2^2+W_2W_3^2+W_4W_2^2))$<br>$\Phi_{SO(k)}(W_5W_3)$<br>$\Phi_{SO(k)}W_2W_3^2)$ |

According to this table, we can see that $F^*$ is an isomorphism when dimension is less than $k+8$ and $F^*$ is a monomorphism in dimension $k+8$.

For $\mathbb{F}_p$-cohomology on $MSO(k)$ with odd prime $p$, we have the following theorem.

**The $\mathbb{F}_3$-cohomology case**: Note that the factor $K(\mathbb{Z}/2, k+5)$ is trivial. Thus we have the following diagram:

| Dimension | Generators in $H^*(Y;\mathbb{F}_3)$ | Image of $F^*$ in $H^*(MSO(k);\mathbb{F}_3)$ |
|---|---|---|
| k | $\iota$ | $\Phi_{SO(k)}$ |
| k+4 | $St_3^4\iota$ | $\Phi_{SO(k)}P_4$ |
| k+8 | $St_3^8\iota$ | $\Phi_{SO(k)}(P_4^2+2P_8)$ |

**The $\mathbb{F}_5$-cohomology case**: Similarly to the $\mathbb{F}_5$ case, we have

| Dimension | Generators in $H^*(Y;\mathbb{F}_3)$ | Image of $F^*$ in $H^*(MSO(k);\mathbb{F}_3)$ |
|---|---|---|
| k | $\iota$ | $\Phi_{SO(k)}$ |
| k+4 | $\nu$ | $\Phi_{SO(k)}P_4$ |
| k+8 | $St_5^8\iota$ | $\Phi_{SO(k)}(P_4^2+2P_8)$ |

**The $\mathbb{F}_p$-cohomology case for** $p > 5$:

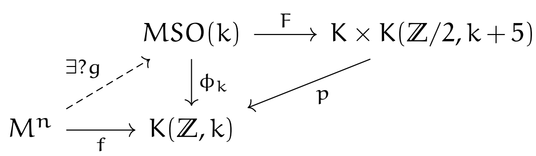
| Dimension | Generators in $H^*(Y;\mathbb{F}_3)$ | Image of $F^*$ in $H^*(MSO(k);\mathbb{F}_3)$ |
|---|---|---|
| k | $\iota$ | $\Phi_{SO(k)}$ |
| k+4 | $\nu$ | $\Phi_{SO(k)}P_4$ |
| k+8 | 0 | 0 |

where the Steenrod operation on $H^*(BSO(k);\mathbb{F}_p)$ can be found in [BS53]. According to these results, we conclude that $F$ is a $k+7$-equivalence. □

**Corollary 3.9** For $i \leqslant 7$, the stable homotopy groups of $MSO(k)$ are presented as the following table:

| $i$ | $\pi_{k+i}(MSO(k))$ |
|---|---|
| 0 | $\mathbb{Z}$ |
| 1,2,3 | 0 |
| 4 | $\mathbb{Z}$ |
| 5 | $\mathbb{Z}/2$ |
| 6,7 | 0 |

With such $(k+7)$-equivalence, we can modify the lifting problem (5) into




(8)

where $p$ is the natural projection to the base space of $K$.

If $k > 8$ and $n < k+8$, an integral cohomology class $x \in H^k(M^n;\mathbb{Z})$ represented by $f\colon M^n \to K(\mathbb{Z},k)$. Then if $St_3^5(x) = 0$, we can lift $f$ to $\widetilde{f}\colon M^n \to K \times K(\mathbb{Z}/2, k+5)$ according to Lemma 3.3. Let $F^{-1}$ be the $(k+7)$-inverse of $F$, then $g := F^{-1} \circ \widetilde{f}$ is the desired answer to the homotopy lifting problem. In summary, we have the following theorem:

**Theorem 3.10** For $k > 8$, an integral $k$-dimensional cohomology class $x$ of a $(k+8)$ dimensional complex is realizable with respect to $SO(k)$ if and only if the integral class $St_3^5(x) = 0$.

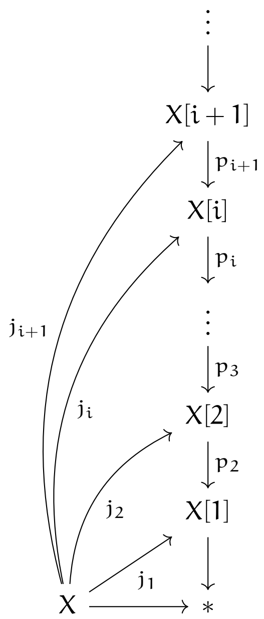
In other words, $St_3^5 x$ is the obstruction of a cohomology class to be realized by an oriented manifold.

# 4 GENERALIZED METHOD: POSTNIKOV TOWER

For general cases of simply connected spaces with more complicated homotopy groups, we just iterate previous construction to obtain a tower consists of such fibrations called **Postnikov tower**.

**Definition 4.1** Let $p\colon X \to Y$ be a fibration with homotopy fiber $F$. Then we say $p$ is a **principal fibration** if there exists a space $B$ with a homotopy equivalent $\Omega B \simeq F$ and a map $f\colon Y \to B$ such that $p\colon X \to Y$ is homotopy equivalent to the pull back of the fibration $PB \to B$ along $f$.

**Theorem 4.2** Suppose $X$ is a path connected space, there exists a commutative diagram




such that

1. $\pi_n X[i] = 0$ for $n > i$;
2. $j_i$ is an $i+1$-equivalence;
3. $p_i\colon X[i] \to X[i-1]$ is a principal fibration with homotopy fiber $K(\pi_i(X), i)$;
4. the natural map $X \to \lim_i X[i]$ is a weak equivalence.

This tower is **Postnikov tower** of $X$ with principal fibrations.

Since the $p_k$ is principal, then we have

$$\begin{array}{ccccc} K(\pi_i(X), i) & \longrightarrow & X[i] & & \\ & & \downarrow{\scriptstyle p_i} & & \\ & & X[k-1] & \xrightarrow{k_{i-1}} & K(\pi_i(X), i+1) \end{array}$$

The cohomology class in $H^{i+1}(X[i-1]; \pi_i(X))$ representing by $k_{i-1}$ is the $i-1$-st $k$**-invariant** of $X$.

**Corollary 4.3** Suppose $X$ is a simply connected space. For any $F\colon Y \to X[i-1]$, there is a uniquely determined class $f^*k_{i-1} \in H^{i+1}(Y, \pi_i(X))$. Then $f$ can be lifted to $X[i]$ along the principal fibration $p_i\colon X[i] \to X[i-1]$ if and only if $f^*k_{i-1} = 0$.

In this way, to solve a homotopy lift problem, we can use the ladders of principal fibrations in the Postnikov tower to exhibit the obstruction of the lifting problem as the pullback of $k$-invariants.

## REFERENCES

[Bro82] Edgar H. Brown, Jr., *The cohomology of* $\mathrm{BSO}_n$ *and* $\mathrm{BO}_n$ *with integer coefficients*, Proc. Amer. Math. Soc. **85** (1982), no. 2, 283–288. MR 652459

[BS53] A. Borel and J.-P. Serre, *Groupes de Lie et puissances réduites de Steenrod*, Amer. J. Math. **75** (1953), 409–448. MR 58213

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. MR 0440554

[Tho54] René Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR 61823