An Introduction to Steenrod Operations

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USTC & SUSTech Meeting

Motivation

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To classify, simplify and compute with geometric objects, algebraic invariants are useful.

Table: Methods of algebraic topology

Geometric objects		Algebraic objects
CW complexes Manifolds Schemes Data sets ...	Chosen invariants $\longrightarrow$	Numbers Groups Rings Chain complexes ...
Geometric Morphisms		Algebraic Morphisms
Homotopy	$\longrightarrow$	Equality

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Example

2-dimensional sphere S^2 is not homotopy equivalent to 2-dimensional torus T^2:

$$H_1(S^2;\mathbb{Z}) = 0 \neq H_1(T^2;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

This means that there is no non-trivial one dimensional hole on S^2 while there are two non-trivial and unequivalent one dimensional hols on T^2.

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Let's compare $\mathbb{CP}^2$ and $S^2 \vee S^4$.

Table: *The homology groups of X and A with $\mathbb{Z}$ coefficient*

	H_0	H_1	H_2	H_3	H_4	$\cdots$
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By considering the cup product structure

$$\smile : H^p(X) \times H^q(X) \longrightarrow H^{p+q}(X)$$

$H^*(X)$ is a commutative graded ring, which gives sharper algebraic pictures than homology groups.

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Back to the case that $\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[u]/(u^3)$ and $S^2 \vee S^4$. We use the following facts to show that they are not homotopy equivalent.

1. $H^*(\mathbb{CP}^2; \mathbb{Z}) = \mathbb{Z}[u]/(u^3)$, where u is a generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$. In particular, $u^2 = u \smile u$ generates $H^4(\mathbb{CP}^2; \mathbb{Z})$.
2. The cup product structure on $H^*(S^2 \vee S^4; \mathbb{Z})$ is trivial, namely, $u \smile v = 0$ for any two cohomology class u, v.
3. $H^*(\mathbb{CP}^2; \mathbb{Z})$ is never isomorphic to $H^*(S^2 \vee S^4; \mathbb{Z})$.

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Blindness of cohomology theory

Due to the cup product structure, cohomology theory present sharper algebraic picture than homology theory. However, there are some spaces with trivial cup product structure.

Proposition

For any space X, the cup product structure on $H^*(\Sigma X)$ is trivial.

We cannot distinguish $\Sigma\mathbb{CP}^2$ and $\Sigma(S^2 \vee S^4) = S^3 \vee S^5$ by cohomology theory! It is blindness of cohomology.

Goal: Cure the blindness.

Method: Construct more invariants.

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The use of cohomology operations

Definition (Cohomology operations)

Let n, m be two integers and G, π be two abelian groups. A cohomology operation T of type $(n, G; m, \pi)$ is a collection of functions $\{T_X\}$ for each space X

$$T_X \colon H^n(X; G) \longrightarrow H^m(X; \pi)$$

such that for any continuous mapping $f \colon X \to Y$, we have $f^*T_Y = T_X f^*$.
Let $\Sigma^* \colon H^n(X) \cong H^{n+1}(\Sigma)$ be the suspension isomorphism. If $\Sigma^* T = T \Sigma^*$, then T is a **stable** *cohomology operation.*

1. $\mathcal{O}(n, G; m, \pi)$: the collection of cohomology operations of type $(n, G; m, \pi)$.
2. $Stab(n, G; m, \pi)$: the collection of stable cohomology operations of type $(n, G; m, \pi)$.

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The simplest non-trivial cohomology operations are **Steenrod squares**(a special case of Steenrod operations).

Definition (Steenrod squares)

Steenrod squares is a collection of stable cohomology operations

$$Sq^i\colon H^q(X;\mathbb{F}_2) \longrightarrow H^{q+i}(X;\mathbb{F}_2),\ \forall q, i \geq 0$$

that satisfying the following properties:

1. Sq^i *are consists of group homomorphisms,*
2. $Sq^0 = id$*,*
3. $Sq^n u = u \smile u$ *if* $\dim u = n$*,*
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5. $Sq^i(u \smile v) = \sum_{i=0}^{j} Sq^i u \smile Sq^{j-i} v$ *(Cartan's formula).*

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Theorem (Serre)

There exists cohomology operations that satisfying all the properties of Steenrod squares and they are unique.

Sketch proof:

1. $H^n(X;G) = [X, K(G,n)]$, where G is an abelian group and $K(G,n)$ is the Eilenberg-Maclane space.
2. By Yoneda lemma, there is an one-one correspondence between $\mathcal{O}(n, G; m, \pi)$ and $H^m(K(\pi, n); G)$.
3. For any abelian group G and suitable integer n, there is a fibration where the total space is contractible, the based space is $K(G,n)$ and the fiber is $K(G, n-1)$.
4. Use Leray-Serre spectral sequence and transgressions to prove the theorem.

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Let's back to the example of $\Sigma\mathbb{CP}^2$ and $S^3 \vee S^5$:

Suppose there is a homotopy equivalence $f\colon S^3 \vee S^5 \to \Sigma\mathbb{CP}^2$, then we consider the Steenrod square

$$Sq^2\colon H^3(\Sigma\mathbb{CP}^2;\mathbb{Z}) \longrightarrow H^5(\Sigma\mathbb{CP}^2;\mathbb{Z})$$

Let u be a generator of $H^2(\mathbb{CP}^2;\mathbb{Z})$, then by the suspension isomorphism, $\Sigma^* u$ is a generator of $H^3(\Sigma\mathbb{CP}^2;\mathbb{Z})$.According to the definition, $Sq^2\Sigma^* u = \Sigma^* Sq^2 u = \Sigma^*(u^2) \neq 0$, a generator of $H^5(\Sigma\mathbb{CP}^2;\mathbb{Z})$.

However, $f^* Sq^2 \Sigma^* u = \Sigma^* Sq^2 f^* u = \Sigma^*(f^*u)^2 = 0$, which leads to contradiction!

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Construction of Steenrod squares

Recall the construction of cup product. Suppose X is a complex, the diagonal map

$$\begin{array}{rccc} D: & X & \longrightarrow & X \times X \\ & x & \longmapsto & (x, x) \end{array}$$

induces

$$D^*: C^*(X) \otimes C^*(X) \simeq C^*(X \times X) \longrightarrow C^*(X)$$

where the equivalence $\simeq$ is given by Eilenberg-Zilber.
To compute cup product, we need to compute

$$D_*: C_*(X) \longrightarrow C_*(X) \otimes C_*(X)$$

Construction of Steenrod squares

If D is a simplicial map, then D_* is clearly. However, D is NOT simplicial for any non-trivial case.

By Alexander-Whitney approximation, there is a simplicial map D_0 such that $D \simeq D_0$ and specifically

$$D_0 \colon [v_0, \dots, v_n] \mapsto \sum_{p=0}^{n} [v_0, \dots, v_p] \times [v_p, \dots, v_n]$$

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Construction of Steenrod squares

Let $\mathbb{F}_2$ act on $X \times X$ by

$$\begin{array}{rccc} T: & X \times X & \longrightarrow & X \times X \\ & (x, y) & \longmapsto & (y, x) \end{array}$$

Observation: $TD = D$ while $TD_0 \neq D_0$.

Construction of Steenrod squares

Fact: There is no T-invariant simplicial approximation for the non-trivial diagonal map.

Problem: Lack of symmetry.

Idea: Measure the derivation from the symmetry.

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Construction of Steenrod squares

1. Although TD_0 and D_0 are not strictly equal, they are homotopic.
2. Let D_1 be the chain homotopy from D_0 to TD_0, where D_1 carries some information of derivation from the symmetry.
3. D_1 is not T-invariant, there is still some derivation from symmetry.
4. $D_1 + TD_1$ is a chain homotopy from D_0 to itself.
5. By acyclic carrier theorem, $D_1 + TD_1$ is homotopic to the constant homotopy of D_0.
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Construction of Steenrod squares

We have a collection of chain homotopies $\{D_i\}_{n\geq 0}^{\infty}$ and they are higher homotopies.

Definition (cup-i product)

For $u \in C^p(X)$ and $v \in C^q(X)$, we define the cup-i product by

$$u \smile_i v \cdot \sigma := u \otimes v \cdot D_i(\sigma)$$

for each $p+q-i$ cell σ.
In particular, when $i = 0$, $\smile_0$ is cup product.

Remark

Cup-i products are the higher version of cup product. If we just consider the cup product, we will loss the higher information.

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Proposition (Coboundary formula)

With $\mathbb{F}_2$ coefficients, we have

$$\delta(u \smile_i v) = u \smile_{i-1} v + v \smile_{i-1} u + \delta u \smile_{i-1} v + u \smile_i \delta v$$

Corollary

With $\mathbb{F}_2$ coefficients, if u is a p dimensional cocycle, then $u \smile_j u$ is a $2p - j$ dimensional cocycle for each j.

Theorem

We define

$$\begin{aligned} Sq^i : \quad H^p(X;\mathbb{F}_2) &\longrightarrow H^{p+i}(X;\mathbb{F}_2) \\ u &\longmapsto u \smile_{p-i} u \end{aligned}$$

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The Steenrod algebra

Definition (The Steenrod algebra)

Let $Stab(r; \mathbb{F}_2)$ be the collection of stable cohomology operations of type $(n, \mathbb{F}_2; n+r, \mathbb{F}_2)$. Then we define a graded $\mathbb{F}_2$ algebra by

$$\mathbb{A}_2 := \bigoplus_{r \geq 0} Stab(r, \mathbb{F}_2)$$

where the multiplication is given by composition.

Theorem

$\mathbb{A}_2$ is generated by Steenrod squares $\{Sq^i\}_{i \geq 0}$.

Cohomology groups $\rightsquigarrow$ Cohomology ring $\rightsquigarrow$ Graded $\mathbb{A}_2$-module

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Further topic

To explain the phenomenon more clearly, we need **operad theory**. For example, there is an E_∞-operad $\mathcal{C}$ where $\mathcal{C}(2)$ encode all D_i. The structure is called E_∞ algebra.

Theorem (May)

Every E_∞ algebra has Steenrod operations naturally.

Theorem (Mandell)

The singular cochain complex of a space is an E_∞ algebra in a canonical way.

Theorem (Mandell)

Suppose X, Y are simply connected spaces, a continuous map $f : X \to Y$ induces a quasi-isomorphism between $C^*(Y)$ and $C^*(X)$ as E_∞ algebra, if and only if f is a weak homotopy equivalence.

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Thank you.