The Steenrod Operations Encode the Data of Homotopy Coherence

Tongtong Liang

Southern University of Science and Technology, China (SUSTech)

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Outline

Why we need algebraic invariants

Definition

Suppose X, Y are two topological spaces, we say X is homeomorphic to Y if there exists two bijective continuous map $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are identities.

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The method of algebraic topology is to assign **algebraic invariants** to each space. Algebraic invariants can be numbers, groups, rings and more complicated algebraic structures.

The use of algebraic invariants

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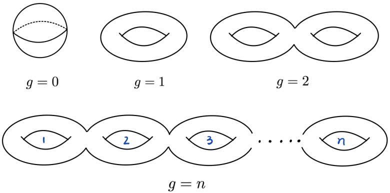
Example (Klein)

2-dimensional closed oriented surfaces can be classified by genus, namely, two closed oriented surfaces are homeomorphic if and only if they have the same genus g.



More examples of algebraic invariants

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We cannot distinguish $\Sigma\mathbb{CP}^2$ and $S^3 \vee S^5$ just by cohomology with cup products,but we can distinguish them by Steenrod operations.

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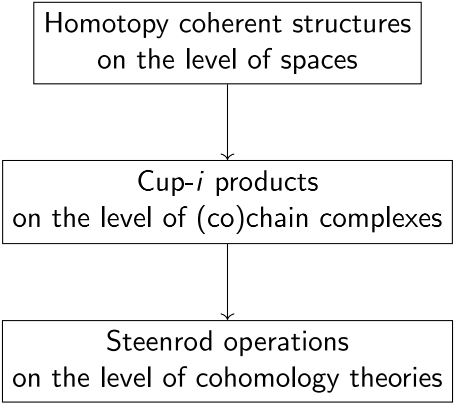
The idea of the thesis



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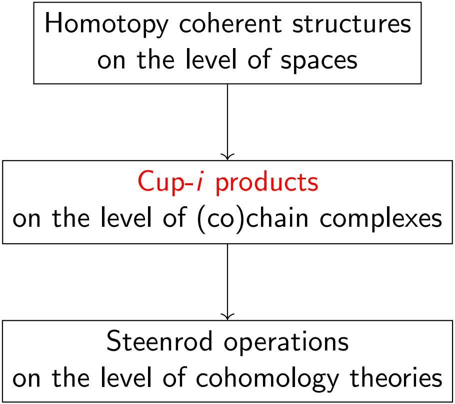

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The notion of homotopy

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A homotopy between two continuous maps $f, g \colon X \to Y$ *is a continuous map* $H \colon X \times I \to Y$ *such that* $H(x, 0) = f(x)$ *and* $H(x, 1) = g(x)$*. We denote it* $f \simeq g$.

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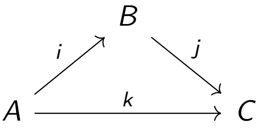
Proposition

A homotopy from f *to* g *is a path in* $\mathrm{Maps}(X, Y)$*, vice versa.*

Homotopy coherent structures

A diagram



is commutative, if $j \circ i = k$; is homotopy commutative, if $j \circ i \simeq k$

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Given a homotopy commutative diagram $\mathcal{D}$, can we find a commutative diagram $\mathcal{D}'$ such that $\mathcal{D} \simeq \mathcal{D}'$ to realize it?

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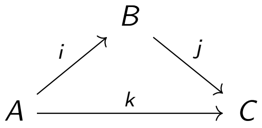

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Theorem (Dwyer-Kan-Smith,1989)

A homotopy commutative diagram has a realization of and only if it may be lifted to a homotopy coherent *diagram.*

Homotopy coherent structures

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n-homotopies $\rightsquigarrow$ n-simplex in the mapping spaces

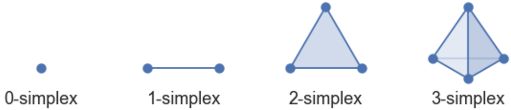

Figure: Simplices

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A category enriched by simplicial sets is a simplicial category.

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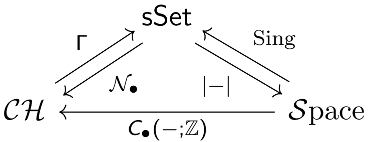

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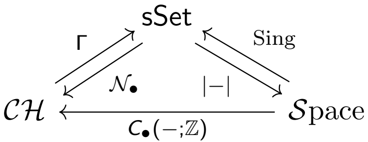

They are $(\infty, 1)$-categories in the sense of Bergner model structure.

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From cup products to cup-i products

$$C_\bullet(X) \xrightarrow{D_0} C_\bullet(X) \otimes C_\bullet(X) \circlearrowleft T \qquad (1)$$

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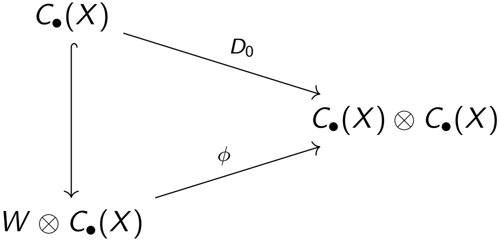
$$\begin{array}{ccc} W \otimes C_\bullet(X) & \xrightarrow{\phi} & C_\bullet(X) \otimes C_\bullet(X) \\ \downarrow{\scriptstyle T\otimes \mathrm{id}} & & \downarrow{\scriptstyle T} \\ W \otimes C_\bullet(X) & \xrightarrow{\phi} & C_\bullet(X) \otimes C_\bullet(X) \end{array} \tag{2}$$

From cup products to cup-i products

D_0 gives cup products, while ϕ gives cup-i products.



In brief, cup-i products are higher derivation of cup products

$$\smile_i : C^p(X) \times C^q(X) \longrightarrow C^{p+q-i}(X)$$

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The construction of the Steenrod operations

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Take $\mathbb{F}_2$-coefficients, suppose $[u]$ is an n-dimensional cohomology class of $H^*(X;\mathbb{F}_2)$, the Steenrod square is defined by

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Remark

The Steenrod squares can be computed by spectral sequences, which are given by the fibrations of Elienberg-Maclane spaces.

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Summary

$$\mathcal{S}\mathrm{pace}_\infty \qquad\qquad \mathcal{CH}_\infty$$

$$\begin{array}{ccc} \mathrm{h}\mathcal{S}\mathrm{pace} & \xrightarrow{H^*} & \mathsf{CDGA} \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle \pi} \\ \mathcal{S}\mathrm{pace} & \xrightarrow{C^\bullet} & \mathcal{CH} \end{array}$$

CDGA: commutative differential graded algebra;
$\mathcal{S}\mathrm{pace}_\infty$: $(\infty, 1)$-category of spaces;
$\mathcal{CH}_\infty$: $(\infty, 1)$-category of chain complexes.

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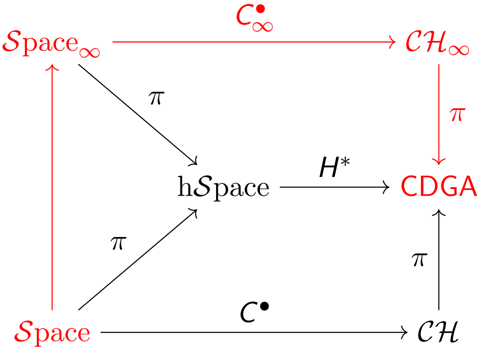

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Question&Answer

Thank you!