Power Operations in Ordinary Cohomology

Tongtong Liang

Southern University of Science and Technology, China (SUSTech)

June 12, 2021

Outline

An Overview of Power Operations

Let X be a CW-complex, the diagonal map from

$$X \xrightarrow{D} \underbrace{X \times \cdots \times X}_{m} = X^m$$

Let the symmetry group of m letters Σ_m act on X^m by permuting the factors and let D_0 be the cellular approximation of D, we have the following homotopy-commutative diagram

$$X \xrightarrow{D_0} X^m \circlearrowright \Sigma_m$$

Note that this diagram is not strictly commutative.

An Overview of Power Operation

There is a strictly commutative diagram as equivalent replacement of the previous diagram.

$$\Sigma_m \circlearrowright E\Sigma_m \times X \xrightarrow{\phi_m} X^m \circlearrowleft \Sigma_m$$

where $E\Sigma_m \to B\Sigma_m$ is the universal principal Σ_m-bundle. Let Σ_m act on X trivially and act on the L.H.S diagonally, then ϕ_m is Σ_m-equivariant.

An Overview of Power Operations

By quotient the group action, we have

$$B\Sigma_m \times_{\Sigma_m} X = E\Sigma_m \times X//\Sigma_m \xrightarrow{P_m} X$$

By passing to cohomology, we have

$$H^*(X) \xrightarrow{\mathcal{P}_m} H^*(X) \otimes H^*(B\Sigma_m)$$

which is called **total** m**-power operation**.

Example: the total Steenrod square

Example

Let $m = 2$, $\Sigma_2 = \mathbb{Z}/2$, and we take $\mathbb{F}_2$-coefficient cohomology, then the total 2-power operation is the total Steenrod squares

$$\begin{array}{rccc} Sq: & H^*(X;\mathbb{F}_2) & \longrightarrow & H^*(X;\mathbb{F}_2)\otimes_{\mathbb{F}_2}\mathbb{F}_2[t] \\ & u & \longmapsto & \sum Sq_i(u)t^i \end{array}$$

Note that $B\mathbb{Z}/2 = \mathbb{RP}^\infty$ and $H^*(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[t]$, where $\dim t = 1$. For $\dim u = n$, $Sq_i = u \smile_i u = Sq^{n-i}u$. We may also write it into a ring homomorphism.

$$Sq: H^*(X;\mathbb{F}_2) \longrightarrow H^*(X;\mathbb{F}_2) \qquad u \longmapsto \sum Sq^i(u)$$

Digression: E_∞-structure on the cochain complexes

Construction

$\{C_\bullet(E\Sigma_m)\}_{m\geq 0}$ forms an E_∞-operad in the category of chain complexes naturally. Then for each m, we can define

$$\theta_m\colon C_\bullet(E\Sigma_m)\otimes C^\bullet(X)^{\otimes m}\to C^\bullet(X)$$

by $\theta_m(\xi\otimes u_1\otimes\cdots\otimes u_m)\cdot\sigma := (u_1\otimes\cdots\otimes u_m)\cdot P_{m\bullet}(\sigma\otimes\xi)$. When $m=2$, this gives cup-i products.

Key questions

Question

What is $EG \to BG$? What properties does it have?

Key questions

Question

What is $EG \to BG$? What properties does it have?

Question

How to construct the Σ_m-equivariant simplicial/cellular map

$$\Sigma_m \circlearrowright E\Sigma_m \times X \xrightarrow{\phi_m} X^m \circlearrowleft \Sigma_m$$

Outline

Milnor construction of universal bundles

Definition (The Milnor join)

Let $\{X_j \mid j \in J\}$ be a family of spaces, the join

$$X = \bigstar_{j \in J} X_j$$

is defined as follows. Each element in X can be represented by

$$(t_j x_j \mid j \in J), \quad t_j \in [0,1], \quad x_j \in X_j, \quad \sum_{j \in J} t_j = 1$$

with only finitely many non-zero t_j. $(t_j x_j)_{j \in J} \sim (t'_j x'_j)_{j \in J}$ if and only $t_j x_j = t'_j x'_j$ for all $t_j \neq 0$, where one may take $t_j x_j$ as (t_j, x_j).

Milnor construction of universal bundles

Definition (The Milnor topology of X)

The Milnor topology of X is the coarsest topology such that the following maps are continuous

$$t_j\colon X \to [0,1],\ (t_i x_i) \mapsto t_j \quad p_j\colon t_j^{-1}([0,1]) \to X_j,\ (t_i x_i) \mapsto x_j$$

Milnor construction of universal bundles

Definition (The Milnor topology of X)

The Milnor topology of X is the coarsest topology such that the following maps are continuous

$$t_j \colon X \to [0,1],\ (t_i x_i) \mapsto t_j \quad p_j \colon t_j^{-1}([0,1]) \to X_j,\ (t_i x_i) \mapsto x_j$$

Example

Milnor construction of universal bundles

Definition (The Milnor topology of X)

The Milnor topology of X is the coarsest topology such that the following maps are continuous

$$t_j\colon X \to [0,1],\ (t_i x_i) \mapsto t_j \quad p_j\colon t_j^{-1}([0,1]) \to X_j,\ (t_i x_i) \mapsto x_j$$

Example

- $\{*\} \star X \simeq CX$;

Milnor construction of universal bundles

Definition (The Milnor topology of X)

The Milnor topology of X is the coarsest topology such that the following maps are continuous

$$t_j\colon X \to [0,1],\ (t_i x_i) \mapsto t_j \quad p_j\colon t_j^{-1}([0,1]) \to X_j,\ (t_i x_i) \mapsto x_j$$

Example

- $\{*\} \star X \simeq CX$;
- $\{0\} \star \{1\} = [0,1]$;

Milnor construction of universal bundles

Definition (The Milnor topology of X)

The Milnor topology of X is the coarsest topology such that the following maps are continuous

$$t_j\colon X \to [0,1],\ (t_i x_i) \mapsto t_j \quad p_j\colon t_j^{-1}([0,1]) \to X_j,\ (t_i x_i) \mapsto x_j$$

Example

- $\{*\} \star X \simeq CX$;
- $\{0\} \star \{1\} = [0,1]$;
- $\Delta^n \star \Delta^m = \Delta^{n+m+1}$

Milnor construction of universal bundle

Construction

*Suppose G is a group, the **Milnor space** is*

$$EG := G \star G \star G \star G \star \dots$$

a join of countably infinitely many copies of G. Let G act on EG by $g(t_j g_j) = (t_j g g_j)$.

Milnor construction of universal bundle

Construction

*Suppose G is a group, the **Milnor space** is*

$$EG := G \star G \star G \star G \star \dots$$

a join of countably infinitely many copies of G. Let G act on EG by $g(t_j g_j) = (t_j g g_j)$.

Proposition

EG is a G-space with free action.

Milnor construction of universal bundle

Construction

*Suppose G is a group, the **Milnor space** is*

$$EG := G \star G \star G \star G \star \dots$$

a join of countably infinitely many copies of G. Let G act on EG by $g(t_j g_j) = (t_j g g_j)$.

Proposition

EG is a G-space with free action.

Proposition

EG is contractible.

Properties of the Milnor space

Proposition

Let E be a G-space, then any two G-maps $f, g \colon E \to EG$ are G-homotopic.

Properties of the Milnor space

Proposition

Let E be a G-space, then any two G-maps $f, g\colon E \to EG$ are G-homotopic.

Sketch proof.

We may write $f(x)$ and $g(x)$ into the following forms

$$f(x) = (t_1(x)f_1(x), t_2(x)f_2(x), t_3(x)f_3(x), \dots)$$
$$g(x) = (u_1(x)g_1(x), u_2(x)g_2(x), u_3(x)g_3(x), \dots)$$

Then prove $(t_1f_1, t_2f_2, t_3f_3, \dots) \sim_G (t_1f_1, 0, t_3f_3, 0 \dots)$. Similarly, $(u_1g_1, u_2g_2, u_3g_3, u_4g_4 \dots) \sim_G (0, u_2g_2, 0, u_4g_4 \dots)$. Finally, prove these maps are G-homotopic to $(t_1f_1, u_2g_2, t_3f_3, u_4g_4 \dots)$ □

Properties of the Milnor space

Corollary

EG is contractible.

Properties of the Milnor space

Corollary

EG is contractible.

Corollary

If E is a contractible G-free space, then E is G-homotopic equivalent to EG.

Properties of the Milnor space

Corollary

EG is contractible.

Corollary

If E is a contractible G-free space, then E is G-homotopic equivalent to EG.

Definition (Classifying space)

The classifying space BG is defined by EG/G. The quotient map $EG \to BG$ is a fiber bundle with fiber G.

Δ-structure of EG

Construction

Let n-simplices be ordered $(n+1)$-tuples $[g_0, \ldots, g_n]$ of G representing

$$\{\sum_{i=0}^{n} a_i g_i \in EG \mid \sum_{i=0}^{n} a_i = 1,\ a_i \in [0,1]\}$$

and the i-face of $[g_0, \ldots, g_n]$ is $[g_0, \ldots, \hat{g}_i, \ldots, g_n]$. In this way, EG is a Δ-complex and each automorphism given by G on EG is a simplicial map.

Simplicial structure of EG

Construction

We define a simplicial set E_*G by setting

$$E_nG := G^{n+1}$$

where $s_i \colon G^n \to G^{n+1}$ is adding identity at i and $d_j \colon G^{n+1} \to G^n$ is merging g_j and g_{j+1} by group operation. Let G acts on G^m on the left

$$g(g_0, g_1, \ldots, g_n) = (gg_0, gg_1, \ldots, gg_n)$$

then E_*G is a simplicial free G-space.

Simplicial structure of EG

Proposition

The geometric realization of E_*G is exactly EG

$$EG = |E_*G| = \coprod_{n\geq 0} G^{n+1} \times \Delta^n / \sim$$

Simplicial structure of EG

Proposition

The geometric realization of E_*G is exactly EG

$$EG = |E_*G| = \coprod_{n \geq 0} G^{n+1} \times \Delta^n / \sim$$

Remark

The simplicial set E_*G is the homotopy coherent nerve of the simplicial resolution of the groupoid $\mathcal{B}G$ of G.

Outline

Total m-power operations

Goal

Prove that there is a Σ_m-equivariant simplicial(or cellular) map $\phi\colon E\Sigma_m \times X \to X^m$ which is equivalent to the diagonal map.

Total m-power operations

Goal

Prove that there is a Σ_m-equivariant simplicial(or cellular) map $\phi\colon E\Sigma_m \times X \to X^m$ which is equivalent to the diagonal map.

Observation

If such ϕ exists, then we have a simplicial map $X \to X^m$ given by

$$\begin{array}{ccc} [e] \times X & & \\ \downarrow & \searrow^{D_0} & \\ E\Sigma_m \times X & \xrightarrow{\phi} & X^m \end{array}$$

and it should be homotopic to the diagonal map.

Power operations encode homotopy coherence

Observation

Since ϕ is Σ_m-equivariant, for any $g \in \Sigma_m$, we have

$$\begin{array}{ccc} [g] \times X & & \\ \big\downarrow & \searrow^{gD_0} & \\ E\Sigma_m \times X & \xrightarrow{\ \phi\ } & X^m \end{array}$$

Power operations encode homotopy coherence

Observation

Given any two elements g, g' in Σ_m, gD_0 is homotopic to $g'D_0$ by

$$\begin{array}{ccc} [g,g'] \times X & & \\ \big\downarrow & \searrow^{h_{g,g'}} & \\ E\Sigma_m \times X & \xrightarrow{\ \phi\ } & X^m \end{array}$$

Power operations encode homotopy coherence

Observation

Given there elements g_0, g_1, g_2, let $h_{i,j}$ be the homotopy from $g_i D_0$ to $g_j D_0$ for $0 \geq i < j \geq 2$, then the join homotopy $h_{0,1} * h_{1,2}$ is homotopic to $h_{0,2}$, which is performed by

$$\begin{array}{ccc} [g_0, g_1, g_2] \times X & & \\ \downarrow & \searrow^{h_{0,1,2}} & \\ E\Sigma_m \times X & \xrightarrow{\phi} & X^m \end{array}$$

Power operations encode homotopy coherence

According to these observations, we conclude that such $EG \times X \to X^m$ if and only if the higher homotopies exist and fit together coherently. Intuitively speaking, an n-simplex in EG corresponds to n-homotopies from X to X^m.

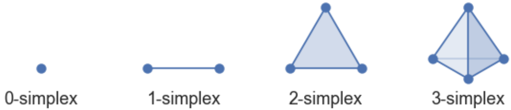

Figure: Simplices

Algebraic argument on chain complexes

Now our goal is to show the existence of higher coherent homotopies. By using methods of algebraic topology, we convert this topology problem to an algebraic problem.

Table: Methods of algebraic topology

Geometric world	Algebraic world
CW complexes	Cellular complexes
Δ-complexes	Simplicial complexes
Cellular maps	chain maps
Simplicial maps	chain maps
homotopies	chain homotopies

Algebraic argument on chain complexes

Steps of the argument:

1. Show the existence of chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$;

Algebraic argument on chain complexes

Steps of the argument:

1. Show the existence of chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$;
2. Show $\{gD_0\}_{g \in \Sigma_m}$ are homotopic;

Algebraic argument on chain complexes

Steps of the argument:

1. Show the existence of chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$;
2. Show $\{gD_0\}_{g \in \Sigma_m}$ are homotopic;
3. Show the existence of higher chain homotopies.

Algebraic argument on chain complexes

Steps of the argument:

1. Show the existence of chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$;
2. Show $\{gD_0\}_{g\in\Sigma_m}$ are homotopic;
3. Show the existence of higher chain homotopies.

Remark

For convenience, we identify X and $C_\bullet(X)$ (when X is a CW-complex, $C_\bullet(X)$ is its cellular chain complex; when X is a Δ-complex, $C_\bullet(X)$ is its simplicial chain complex.

Carriers between complexes

Definition (Carrier)

*A **carrier** from complexes pair (K, L) to (K', L') is a function which assigns to each simplex (or cell) σ of K a non-trivial subcomplex $\mathcal{C}(\sigma)$ of K such that $\sigma \in L$ implies $\mathcal{C}(\sigma) \subset L'$ and if $\tau < \sigma$ (this means τ is a face of σ), then $\mathcal{C}(\tau) \subset \mathcal{C}(\sigma)$. **A carrier is acyclic, if $\mathcal{C}(\sigma)$ is acyclic for each simplex $\sigma \in K$.***

Carriers between complexes

Definition (Carrier)

*A **carrier** from complexes pair (K, L) to (K', L') is a function which assigns to each simplex (or cell) σ of K a non-trivial subcomplex $\mathcal{C}(\sigma)$ of K such that $\sigma \in L$ implies $\mathcal{C}(\sigma) \subset L'$ and if $\tau < \sigma$ (this means τ is a face of σ), then $\mathcal{C}(\tau) \subset \mathcal{C}(\sigma)$. **A carrier is acyclic, if $\mathcal{C}(\sigma)$ is acyclic for each simplex $\sigma \in K$.***

Remark

*We say **a carrier carries a chain homotopy** h if for each cell σ, $h(\sigma) \in \mathcal{C}(\sigma)$. Similarly, **a carrier carries a chain map** ϕ if $\phi(\sigma) \in \mathcal{C}(\sigma)$.*

Carriers between complexes

Let σ be a simplex in X, then let $\bar{\sigma}$ be the subcomplex of X that consists of all the faces of σ. If we identify σ as an embedding of $\sigma\colon \Delta^n \to X$, then $\bar{\sigma} = \mathrm{im}\{\sigma_{\#}\colon C_\bullet(\Delta^n) \to C_\bullet(X)\}$.

Example

Now we define a carrier $\mathcal{C}$ from X to X^m by $\mathcal{C}(\sigma) := \underbrace{\bar{\sigma} \otimes \cdots \otimes \bar{\sigma}}_{m}$ *called* ***symmetric carrier***.

Carriers between complexes

Let σ be a simplex in X, then let $\bar{\sigma}$ be the subcomplex of X that consists of all the faces of σ. If we identify σ as an embedding of $\sigma\colon \Delta^n \to X$, then $\bar{\sigma} = \operatorname{im}\{\sigma_{\#}\colon C_\bullet(\Delta^n) \to C_\bullet(X)\}$.

Example

*Now we define a carrier $\mathcal{C}$ from X to X^m by $\mathcal{C}(\sigma) := \underbrace{\bar{\sigma} \otimes \cdots \otimes \bar{\sigma}}_{m}$ called **symmetric carrier**.*

Remark

The concept of carriers makes sense in topological consideration and the carrier in the example carries the diagonal map clearly. If we take σ as a cell, it also makes sense.

Acyclic carrier lemma

Lemma (Acyclic carrier lemma)

If $\mathcal{C}$ is an acyclic carrier $K \to L$, then $\mathcal{C}$ carries a chain map ϕ; and, if ϕ, ψ are two chain maps carried by $\mathcal{C}$, then ϕ is homotopic to ψ.

Acyclic carrier lemma

Lemma (Acyclic carrier lemma)

If $\mathcal{C}$ is an acyclic carrier $K \to L$, then $\mathcal{C}$ carries a chain map ϕ; and, if ϕ, ψ are two chain maps carried by $\mathcal{C}$, then ϕ is homotopic to ψ.

Sketch proof.

We may construct the chain map inductively. First, it is easier to construct a suitable morphism $C_0(K) \to C_0(L)$. Then use the acyclicness to construct map $C_n(K) \to C_n(L)$ according to $C_{n-1}(K) \to C_{n-1}(L)$. Similarly, we can construct the required chain homotopy in this way. □

The use of acyclic carrier lemma

1. By using the acyclic carrier lemma, we have shown that there is a chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$ carried by the symmetric carrier $\mathcal{C}$. In this way, we have $\Sigma_m \times X \to X^m$;

The use of acyclic carrier lemma

1. By using the acyclic carrier lemma, we have shown that there is a chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$ carried by the symmetric carrier $\mathcal{C}$. In this way, we have $\Sigma_m \times X \to X^m$;
2. $\{gD_0\}_{g \in \Sigma_m}$ are also carried by $C(\sigma)$, hence they are homotopic and the homotopies are carried by $\mathcal{C}$. In this way, we have $(\Sigma_m \star \Sigma_m) \times X \to X^m$ to exhibit the homotopies;

The use of acyclic carrier lemma

1. By using the acyclic carrier lemma, we have shown that there is a chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$ carried by the symmetric carrier $\mathcal{C}$. In this way, we have $\Sigma_m \times X \to X^m$;
2. $\{gD_0\}_{g \in \Sigma_m}$ are also carried by $C(\sigma)$, hence they are homotopic and the homotopies are carried by $\mathcal{C}$. In this way, we have $(\Sigma_m \star \Sigma_m) \times X \to X^m$ to exhibit the homotopies;
3. Define a carrier from $(\Sigma_m \star \Sigma_m) \times X$ to $X^{\otimes m}$ by $\mathcal{C}^1(\tau \otimes \sigma) := \mathcal{C}(\sigma)$, then by acyclic carrier lemma, we have $(\Sigma_m \star \Sigma_m \star \Sigma_m) \times X \to X^m$ to exhibit the 2-homotopies;

The use of acyclic carrier lemma

1. By using the acyclic carrier lemma, we have shown that there is a chain map $D_0 \colon C_\bullet(X) \to C_\bullet(X)^{\otimes m}$ carried by the symmetric carrier $\mathcal{C}$. In this way, we have $\Sigma_m \times X \to X^m$;
2. $\{gD_0\}_{g\in\Sigma_m}$ are also carried by $C(\sigma)$, hence they are homotopic and the homotopies are carried by $\mathcal{C}$. In this way, we have $(\Sigma_m \star \Sigma_m) \times X \to X^m$ to exhibit the homotopies;
3. Define a carrier from $(\Sigma_m \star \Sigma_m) \times X$ to $X^{\otimes m}$ by $\mathcal{C}^1(\tau \otimes \sigma) := \mathcal{C}(\sigma)$, then by acyclic carrier lemma, we have $(\Sigma_m \star \Sigma_m \star \Sigma_m) \times X \to X^m$ to exhibit the 2-homotopies;
4. Inductively, the limit $EG \times X = \lim_{n\to\infty}(\bigstar_n \Sigma_m) \times X \to X^m$ is what we need.

Equivariant acyclic carrier lemma

Definition (Equivariant carriers)

Suppose X, Y are two G-complexes and $\mathcal{C}$ is a carrier from X to Y, we say $\mathcal{C}$ is equivariant if $\mathcal{C}(g \cdot \sigma) = g \cdot \mathcal{C}(\sigma)$, for any $g \in G$ and any cell σ.

Equivariant acyclic carrier lemma

Definition (Equivariant carriers)

Suppose X, Y are two G-complexes and $\mathcal{C}$ is a carrier from X to Y, we say $\mathcal{C}$ is equivariant if $\mathcal{C}(g \cdot \sigma) = g \cdot \mathcal{C}(\sigma)$, for any $g \in G$ and any cell σ.

Lemma (Equivariant acyclic carrier lemma)

Let K' be a G-subcomplex of G-free complex K, and suppose there is an equivariant map from $\phi' \colon K' \to L$ carried by an equivariant acyclic carrier $\mathcal{C}$ from K to L, then we may extend ϕ' to an equivariant map $\phi \colon K \to L$. Any two G-equivariant extensions are G-homotopic.

The use of equivariant acyclic carrier lemma

Example

Define an equivariant carrier from $E\Sigma_m \times X$ to X^m by $\mathcal{C}^e(\tau \times \sigma) := \mathcal{C}(\sigma)$, $\forall \tau \subset E\Sigma_m$, $\forall \sigma \subset X$. Note that we have a Σ_m-equivariant map defined by $\phi_0 \colon G \times X \to X^m$, $g \times x \to gD_0(x)$, then the equivariant acyclic carrier lemma allows an extension

$$\begin{array}{ccc} \Sigma_m \times X & & \\ \big\downarrow & \overset{\phi_0}{\searrow} & \\ E\Sigma_m \times X & \xrightarrow{\phi} & X^m \end{array}$$

(one way regard Σ_m as the 0-skeleton of $E\Sigma$ by $g \mapsto [g]$).

Outline

Digression: the total power operation on K-theory

Example

By passing to complex K-theory, we have

$$K(X) \xrightarrow{\mathcal{P}_m} K(X \times_{\Sigma_m} B\Sigma_m)$$

Do we have $K(X \times_{\Sigma_m} B\Sigma_m) \cong K(X) \otimes R(\Sigma_m)$ *in this way?*

Digression: the total power operation on K-theory

Example

By passing to complex K-theory, we have

$$K(X) \xrightarrow{\mathcal{P}_m} K(X \times_{\Sigma_m} B\Sigma_m)$$

Do we have $K(X \times_{\Sigma_m} B\Sigma_m) \cong K(X) \otimes R(\Sigma_m)$ *in this way?*

Theorem (Atiyah-Segal)

Suppose G is a finite group or compact Lie group, then $R(G)^\vee \cong K(BG)$, where $R(G)^\vee$ is the ***formal completion*** *of the ring $R(G)$ at the* ***augmentation ideal*** *(the augmentation is the character).*