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UNIVERSAL PROPERTY OF K-THEORY AND GROTHENDIECK-RIEMANN-ROCH THEOREM

LIANG TONGTONG

ABSTRACT. This is a survey on the Grothendieck-Riemann-Roch theorem. The main reference is [Nav16]. The idea of this article is to show that K-theory is the universal cohomology theory with multiplicative law $x + y - xy$, then Grothendieck-Riemann-Roch theorem follows the result.

CONTENTS

1. INTRODUCTION

For any smooth projective variety X (or more general, a Noetherian, factorial separated schemea), the Picard group of line bundles $\mathrm{Pic}(X)$ is isomorphic to the Weil divisor class group $\mathrm{Cl}(X)$ via the correspondence

$$D \mapsto \mathscr{O}_X(D).$$

where D is a Weil divisor. The inverse map

$$\mathscr{L} \cong \mathscr{O}_X(D) \mapsto D$$

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is called **the first Chern class** and we usually denote it $c_1(\mathscr{L}) = D$. This correspondence shows the connection between closed subvarieties of codimension 1 and the line bundles.

Then Riemann-Roch theorem shows that for smooth projective curve C and divisor D, we have

$$\chi(C, \mathscr{O}_C(D)) = \chi(C, \mathscr{O}_C) + \deg D.$$

we may write it into a diagram

$$\begin{array}{ccc} \mathrm{Cl}(C) & \underset{c_1}{\overset{D \mapsto \mathscr{O}_C(D)}{\rightleftarrows}} & \mathrm{Pic}(C) \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle \chi} \\ \mathbb{Z} & \xleftarrow[-\chi(\mathscr{O}_C)]{} & \mathbb{Z} \end{array} \tag{1}$$

Let K be the canonical divisor on C, we have $\deg K = 2g - 2 = -2\chi(\mathscr{O}_C)$ by Serre duality and the Riemann-Roch theorem, then we may rewrite the diagram into

$$\begin{array}{ccc} \mathrm{Cl}(C) & \underset{c_1}{\overset{D \mapsto \mathscr{O}_C(D)}{\rightleftarrows}} & \mathrm{Pic}(C) \\ \downarrow{\scriptstyle \deg} & & \downarrow{\scriptstyle \chi} \\ \mathbb{Z} & \xrightarrow[-\frac{1}{2}\deg(K)]{} & \mathbb{Z} \end{array} \tag{2}$$

Note that $\deg : \mathrm{Cl}(X) \to \mathbb{Z}$ is a group homomorphism clearly and the characteristic is additive, namely, if there is a short exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0,$$

we have $\chi(\mathscr{F}_2) = \chi(\mathscr{F}_1) + \chi(\mathscr{F}_3)$. However, the diagram (1) and (2) are not good diagrams, because

(1) χ is not a group homomorphism.
(2) either $-\chi(\mathscr{O}_C)$ or $-\frac{1}{2}\deg(K)$ is not a natural map.

To make it more natural, we need to modify $\mathrm{Pic}(C)$, $\mathrm{Cl}(C)$, c_1, deg and χ.

The first idea is to make the short exact sequences of vector bundles or quasi-coherent sheaves into "addition", and that is what exactly Grothendieck's K-theory $K(X)$ (see section 2.1) does. Then we also need to promote the concept of the class group of Weil divisors, and the concept of Chow group $CH^\bullet(X)$ (see 2.2) is what we need. Hence we expect to modify diagram (1) and (2) to make it like a diagram of natural transformation and it is what the Grothendieck-Riemann-Roch theorem says.

Theorem 1.1 (Grothendieck-Riemann-Roch)**.** *Let $f : Y \to X$ be a projective morphism between smooth quasi-projective algebraic varieties and denote T_X and T_Y their tangent bundle, then we have a commutative diagram:*

$$\begin{array}{ccc} K(Y) & \xrightarrow{f_!} & K(X) \\ \downarrow{\scriptstyle Td(T_Y)\cdot ch} & & \downarrow{\scriptstyle Td(T_X)\cdot ch} \\ CH^\bullet(Y) \otimes \mathbb{Q} & \xrightarrow{f_*} & CH^\bullet(X) \otimes \mathbb{Q} \end{array} \tag{3}$$

Remark 1.2. Let Y be a smooth irreducible projective curve C over an algebraic closed field k and $X = \operatorname{Spec} k$, then the diagram (3) implies the diagram (2).

Further, the Grothendieck's K-theory and the Chow ring have cohomological behavior in some sense. To explain this phenomenon clearly, we give the formulation of cohomology theory in section 2 and the definition refers to [Pan04].

In this article, we will show that Grothendieck's K-theory is the universal cohomology theory where Chern classes of line bundles follows

$$c_1(\mathscr{L}\otimes\mathscr{L}') = c_1(\mathscr{L}) + c_1(\mathscr{L}') - c_1(\mathscr{L})c_1(\mathscr{L}'). \tag{4}$$

where the universal property means that for any cohomology theory A follows (4), there is a unique morphism (a natural transformation with some additional conditions) $\varphi\colon K \to A$ between cohomology theories.

Assumption. All the field k in this article are algebraically closed field. Varieties means separated integral schemes of finite type over k. Subvarieties are always closed and irreducible.

2. Cohomology Theory

Definition 2.1. A **cohomology theory** is a contravariant functor A from the category of smooth quasi-projective varieties over a perfect field k, denoted by $\mathbf{smQProj}_k$ to the category of commutative ring **Cring** with the following data:

(1) For each projective morphism $f : Y \to X$, there is a *functorial morphism of* $A(X)$*-modules* $f_* : A(Y) \to A(X)$ called **direct image**. Namely, the slant part means that $\mathrm{id}_* = \mathrm{id}$ and $(fg)_* = f_*g_*$ and the projective formula $f_*(f^*(x)y) = xf_*(y)$ holds.
(2) For any smooth closed embedding of variety $i\colon Y \to X$, **the fundamental class** $[Y]^A := i_*(1) \in A(X)$.
(3) For any line bundle $L \to X$ on X, the **Chern class** is $c_1^A(L) := s_0^*(s_{0*}(1) \in A(X)$, where $s_0\colon X \to L$ is the zero section. (If one defines a line bundle $\mathscr{L}$ as a coherent sheaf on X, then the total space is $\mathbf{Spec}(\mathrm{Sym}^\bullet\mathscr{L}^*)$)

The data satisfies the following axioms

(1) $A(X_1 \coprod X_2) \cong A(X_1) \oplus A(X_2)$ naturally. Hence $A(\emptyset) = 0$.
(2) For any affine bundle $P \to X$, the ring morphism $A(X) \to A(P)$ is an isomorphism.
(3) For any smooth closed subvariety $i\colon Y \to X$, we have an exact sequence

$$A(Y) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(X\setminus Y)$$

(4) If a morphism $f : \bar{X} \to X$ is transversal to a smooth closed subvariety $i\colon Y \to X$ of codimension d (that is to say $f^{-1}(Y) = \emptyset$ or $\bar{Y} = f^{-1}(Y)$ is a smooth subvariety of $\bar{X}$ of codimension d such that the natural morphism $f^*\mathscr{N}_{Y/X} \to \mathscr{N}_{\bar{Y}/\bar{X}}$ is an isomorphism), then the following diagram commutes

$$\begin{array}{ccc} A(Y) & \xrightarrow{f^*} & A(\bar{Y}) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} \\ A(X) & \xrightarrow{f^*} & A(\bar{X}) \end{array}$$

(5) Let $\pi\colon \mathbb{P}(E) \to X$ be a projective bundle. For any morphism $f\colon Y \to X$, the following diagram commutes

$$\begin{array}{ccc} A(\mathbb{P}(E)) & \xrightarrow{f^*} & A(\mathbb{P}(f^*E)) \\ \pi_* \downarrow & & \downarrow i_* \\ A(X) & \xrightarrow{f^*} & A(Y) \end{array}$$

(6) Let $\pi : \mathbb{P}(E) \to X$ be the projective bundle associated to a vector bundle $E \to X$ of rank $r{+}1$ and $\xi_E \to \mathbb{P}(E)$ be the tautological line bundle. Consider the structure of $A(X)$-module in $A(\mathbb{P}(E))$ defined by the ring morphism $\pi^* : A(X) \to A(\mathbb{P}(E))$, and put $x_E = c_1^A(\xi_E)$, then $1, x_E, \dots, x_E^r$ defined a basis

$$A(\mathbb{P}(E)) = A(X) \oplus A(X)x_E \oplus \cdots \oplus A(X)x_E^r$$

Given cohomology theories A and $\bar{A}$ on smooth quasi-projective varieties over k, a **morphism of cohomology theories** $\phi\colon A \to \bar{A}$ is a natural transformation preserving the direct images. Namely, for any morphism $f : Y \to X$ and any $a \in A(X)$, we have $\varphi(f^*(a)) = f^*(\varphi(a))$

Remark 2.2. A cohomology theory is determined by the contravariant functor and the direct image, because all the data in the definition is determined by them.

a. Given a line bundle $L \to X$ and a morphism $f : Y \to X$, the induced morphism $f \times i\colon f^*L = Y \times_X L \to X \times_X L = L$ is transversal to the zero section $s_0\colon X \to L$. Hence according to Axiom (4), we have

$$f^* s_0^* s_{0*}(1) = \bar{s}_0^*(f \times 1)^* s_{0*}(1) = \bar{s}_0^* \bar{s}_{0*}(1)$$

where $\bar{s}_0$ is the zero section $Y \to Y \times_X L$. Hence we conclude that Chern classes are functorial

$$c_1(f^*L) = f^* c_1(L)$$

b. Let $L_Y \to X$ be the line bundle defined by a smooth closed hypersurface $i\colon Y \to X$ i.e. the dual of the ideal sheaf associated to $i(Y)$ in X. It admits a section $X \to L_Y$ vanishing just on Y and transversal to the zero section $s_0\colon X \to L_Y$. Hence

$$c_1(L_Y) = s^*(s_{0*}(1)) - i_*(i^*(1)) = i_*(1) = [Y] \in A(X) \tag{5}$$

In particular, if $\xi_d \to \mathbb{P}^d$ is the tautological line bundle of the projective space of dimension d, the Chern class of the dual bundle ξ_d^* is just the fundamental class of an hyperplane, $c_1(\xi_d^*) = [\mathbb{P}^{d-1}]$.

In general, L_x will denote a line bundle with the first Chern class $c_1(L_x) = x \in A(X)$, and we say that a cohomology follows the additive group law $x + y$ when $c_1^A(L_x \otimes L_y) = x + y$, that it follows the multiplicative group law $x + y - xy$ when $c_1^A(L_x \otimes L_y) = x + y - xy$, and so on.

2.1. **Example: K-theory.** Let X be a Noetherian scheme. Let $G(X)$ be the free abelian group generated by all the coherent sheaves on X. Let $G'(X)$ be the subgroup of $G(X)$ generated by the elements of the form $[\mathscr{F}_1] - [\mathscr{F}_2] - [\mathscr{F}_3]$ where there is an exact sequence

$$0 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

Let $K'(X) = G(X)/G'(X)$ and we call it **Grothendieck K-group**. We have a subgroup $K(X) \subseteq K'(X)$ by requiring all the elements in $K(X)$ can be represented by locally free sheaves.

Theorem 2.3. *If X is regular, then $K(X) = K'(X)$*

Proof. See [Fal92]. □

Hence when X is regular or smooth over a perfect field k, we define the multiplication by

$$[E] \cdot [F] = [E \otimes F]$$

where E, F are vector bundles on X. Equivalently, in the context of coherent sheaves,

$$[E] \cdot [F] = \sum_{i=0}^{\infty} (-1)^i [Tor_i^{\mathscr{O}_X}(E, F)]$$

For any morphism $f: Y \to X$, the pull-back is defined by

$$f^*[E] = [f^*E]$$

where E is a vector bundle on X and $f^*E = E \times_X Y$.

In this way $K: X \mapsto K(X)$ is a contravariant functor from $\mathbf{smQProj}_k$ to **Cring**.

Then we define the direct image by $f_*[E] = [f_*E]$ for any projective morphism $f: Y \to X$ and E is a vector bundle on Y. Since every projective morphism is proper, the projective formula is true:

$$f_*[E \otimes f^*E] = f_*[E] \cdot [F]$$

We claim that in this way, $K: \mathbf{smQProj}_k \to \mathbf{Cring}$ is a cohomology theory defined before. The proof of the claim can be found in [Fal92].

Note that we define the direct image in the context of $K(X)$ and the notation is i_* and i^*. When it comes to K', for $f: Y \to X$ and any coherent sheaf $\mathscr{F}$ on Y, the directed image of $\mathscr{F}$ via f is

$$\sum (-1)^i [R^i f_* \mathscr{F}] \tag{6}$$

Warning: it is not $f_*\mathscr{F}$ simply!. To avoid the abuse of notation, we let

$$i_!(\mathscr{F}) = \sum (-1)^i [R^i f_* \mathscr{F}]$$

be the direct image in the context of K' and it coincides with the direct image defined in K.

According to (5) and (6), the Chern class of a line bundle $L \to X$ is

$$c_1^K(L) = s_0^!(s_{0!}(1)) = 1 - L^* \in K(X)$$

Note that

$$1 - (L \otimes \bar{L})^* = (1 - L^*) + (1 - \bar{L}^*) - (1 - L^*)(1 - \bar{L}^*)$$

so that

$$c_1^K(L \otimes \bar{L}) = c_1^K(L) + c_1^K(\bar{L}) - c_1^K(L)c_1^K(\bar{L})$$

which means that K-theory follows the multiplicative group law.

2.2. **Example: Chow ring.**

Definition 2.4. Let X be a Noetherian normal scheme. The group of n-cycles, denoted by $Z_n(X)$, is the free abelian group generated by the n-dimensional irreducible subvarieties of X. For each n-dimensional subvariety $V \subset X$, we denote by $[V]$ the corresponding element of $Z_n(X)$. A n-cycle α is an element in $Z_n(X)$, and an algebraic cycle α on X is an element of abelian group $\oplus_n Z_n(X)$.

Definition 2.5. An algebraic cycle α on X is **rationally equivalent to zero**, written as $\alpha \sim 0$, if there are irreducible subvarieties $V_1, \ldots, V_m$ of X and a rational function f_i on each V_i such that

$$\alpha = \sum_{i=1}^{m} \operatorname{div}(f_i)$$

where $\operatorname{div}(f_i)$ is a Weil divisor on X and is a $\mathbb{Z}$-linear combination of codimension 1 irreducible closed subsets of V_i (at the same time, they are irreducible subvarities of X as well).

Remark 2.6. Algebraic cycles that are rationally equivalent 0 forms a group clearly and we denote the subgroup of k-cycles that are rationally equivalent to 0 by $B_k(X)$.

Definition 2.7 (Chow group)**.** The **Chow group** of k-cycles on X, denoted by $A_n(X)$, is

$$A_n(X) := Z_n(X)/B_n(X)$$

The direct sum

$$A_*(X) = \bigoplus_n A_n(X)$$

is called the **Chow group of** X.

Next we show the relation between Chow groups and class groups:

Example 2.8. If X is a Noetherian normal k-variety and $\dim(X) = n$, then $A_{n-1} \cong \mathrm{Cl}(X)$.

Actually, Chow groups is an analogy to singular homology groups in algebraic topology. Informally, we may compare a k-cycle on a scheme X with a k-simplex in a manifold (or generally, topological space) M. However, the given Definition 2.5 of rational equivalence is quit strange if we consider it in algebraic topology. Now we given an equivalent definition of rational equivalence, which makes us think about more algebraic topology.

Definition 2.9. A cycle $\alpha \in Z_k(X)$ is rationally equivalent to zero iff there exists $k+1$ dimensional irreducible varieties $W_1, \ldots, W_m$ of $X \times \mathbb{P}^1$ such that the projective maps $p_i : W_i \to \mathbb{P}^1$ are dominant and

$$\alpha = \sum_{i=1}^{m} (p_i^{-1}(0) - p_i^{-1}(\infty))$$

where $p_i^{-1}(0)$ and $p_i^{-1}(\infty)$ are scheme-theoretic fiber; $0 = [0:1] \in \mathbb{P}^1$ and $\infty = [1:0] \in \mathbb{P}^1$.

The proof of the equivalence between Definition 2.5 and Definition 2.9 is in [Ful98],1.6.

Remark 2.10. Informally speaking, we may view it as a kind of homotopy parametrized by the projective line (in classical algebraic geometry, homotopy is parametrized by the real line $\mathbb{R}$). Specifically, given two k-dimensional irreducible subvarieties V_0, V_1 of X, we say $[V_0]$ is rational equivalent to $[V_1]$, denoted by $V_0 \sim V_1$ if $[V_0]-[V_1] \sim 0$, and according to Definition 2.9, $V_0 \sim V_1$ if and only if there is a $k+1$-irreducible subvariety W of $X \times \mathbb{P}^1$ such that V_0 is the fiber of 0 under the projection to $\mathbb{P}^1$ and V_1 is the fiber of ∞. In this setup, we may regard W as a kind of "homotopy cylinder" between V_0 and V_1.

Next we need to define multiplication law on Chow groups to make it into a commutative ring. The construction of the multiplication is a non-trivial result given by Chow.

Theorem 2.11 (Chow's moving lemma)**.** *Given two algebraic cycles Y, Z on $X \in$ **smQProj**, there exists an algebraic cycle Z' rational equivalent to Z such that Z' and Y intersect properly, namely, transversally.*

Theorem 2.12. *If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the condition:*

If subvarieties A, B of X are generically transversal the

$$[A] \cdot [B] = [A \cap B]$$

This structure makes $A(X)$ into an associative, commutative ring graded by codimension.

This theorem gives the definition the multiplicative law in Chow group, so that we called then Chow ring. The proof of these two theorem can by found in [Eis16].

Notation: $R(X)$ denote the function field of $X \in$ smQProj$_k$.

Now we define the direct image for proper morphisms(including projective morphisms) before defining the pull-back of Chow rings. Suppose $f : Y \to X$ is a proper morphism, V is a subvariety of Y and let $W = f(V)$ a closed subvariety of X. Note that $R(W) \hookrightarrow R(V)$ is a field extension induced by f. If $\dim V = \dim W$, this extension is a finite extension, then we define $\deg(V/W) = [R(V) : R(W)]$ if $\dim V = \dim W$, and $\deg(V/W) = 0$ otherwise. Then the direct image f_* is defined by

$$f_*[V] = \deg(V/W)[W]$$

(this is well-defined and the details of the proof can be found in [Ful98])

Definition 2.13 (Pull-back of Chow ring)**.** Suppose $f : Y \to X$ is any morphism in **smQProj**$_k$, and $V \subset X$ is a subvariety, then we define the pull-back by

$$f^*[V] = pr_{Y*}[Y \times Z] \cdot [\Gamma_f]$$

where Γ_f is the graph of f and pr_Y is the natural projection $Y \times X \to Y$.

More details can be found in [Mur14].

Further, Chow ring forms a cohomology theory in this way, the proof can be found in [Fal92], [Ful98], [Eis16].

3. Chern Classes

Definition 3.1. Let $E \to X$ be a vector bundle of rank r. we define the **Chern classes** $c_n^A(E) \in A(X)$ of E to be the coefficients of the characteristic polynomial

$c(E) = x^r - c_1^A(E)x^{r-1} + \cdots + (-1)^r c_r^A(E)$ of the endomorphism of the free $A(X)$-module $A(\mathbb{P}(E))$ defined by the multiplication by $x_E = c_1^A(\xi_E)$,

$$x_E^r - c_1^A(E)x_E^{r-1} + \cdots + (-1)^r c_r^A(E) = 0$$

and we write $c_n(E)$ when the cohomology theory is clear.

This definition is due to Grothendieck.

Theorem 3.2. *For any morphism $f : Y \to X$, we have $c_n(f^*E) = f^*(c_n(E))$. Namely, Chern classes are functorial.*

Proof. Any morphism $f : Y \to X$ induces a morphism $f : \mathbb{P}(f^*E) \to \mathbb{P}(E)$ such that $f^*\xi_E = \xi_{f^*E}$. Hence $f^*(x_E) = x_{f^*E}$. We apply f^* to the polynomial and we have

$$x_{f^*E}^r - (f^*c_1(E))x_{f^*E}^{r-1} + \cdots + (-1)^r f^*c_r(E) = 0$$

and $c_n(f^*E) = f^*c_n(E)$. □

Theorem 3.3 (Splitting principle)**.** *There exists a base change $\pi : X' \to X$ such that π^*E admits a filtration $0 = E_0 \subset E_1 \cdots \subset E_r = \pi^*(E)$ whose quotients E_i/E_{i-1}, for $i = 1, \ldots, r$, are line bundles, and $\pi^* : A(X) \to A(X)$ is injective.*

Proof. In $\mathbb{P}(E)$, we have an exact sequence

$$0 \longrightarrow \xi_E \longrightarrow \pi^*E \longrightarrow Q \longrightarrow 0$$

where Q is given by the cokernel and is of rank $r - 1$. According to Axiom 6, π^* is injective. Then we may proceed by induction to prove splitting principle. □

Theorem 3.4. *Chern classes are additive. Namely $c(E) = c(E_1)c(E_2)$ for any exact sequence*

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

of vector bundles, then

$$c_n(E) = \sum_{i+j=n} c_i(E_1)c_j(E_2)$$

for $n, i, j \in \mathbb{N}$.

Proof. First, we may assume E_1 is a line bundle. Then the induced map $i : X = \mathbb{P}(E_1) \to \mathbb{P}(E)$ is a section of $\mathbb{P}(E) \to X$, so that i_* is injective.

Let $U = \mathbb{P}(E) \setminus \mathbb{P}(E_1)$ and $j : U \to \mathbb{P}(E)$ be the open embedding. Note that the restriction morphism $p : U \to \mathbb{P}(E_2)$ exhibit U as an affine bundle on $\mathbb{P}(E_2)$ because the fiber is $\mathbb{P}(E_1) \setminus \{*\}$ and $j^*\xi_E = p^*\xi_{E_2}$, so that $j^*(x_E^n) = p^*(x_{E_2}^n)$. Hence $j^* : A(_1) \to A(U) = A(\mathbb{P}(E_2))$ is surjective (by Axiom (2) and Axiom (6)). Then by Axiom (3), we have the following commutative diagram of exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(\mathbb{P}(E_1)) & \xrightarrow{i_*} & A(\mathbb{P}(E)) & \xrightarrow{j^*} & A(\mathbb{P}(E_2)) & \longrightarrow & 0 \\
 & & \downarrow{\cdot x_{E_1}} & & \downarrow{\cdot x_E} & & \downarrow{\cdot x_{E_2}} & & \\
0 & \longrightarrow & A(\mathbb{P}(E_1)) & \xrightarrow{i_*} & A(\mathbb{P}(E)) & \xrightarrow{j^*} & A(\mathbb{P}(E_2)) & \longrightarrow & 0
\end{array}$$

Thus $c(E) = c(E_1)c(E_2)$ by basic fact in linear algebra.

In general case, we may do induction by splitting principle on the rank of E_1. We may assume we have a line bundle $L \subset E_1$ such that $\bar{E}_1 = E_1/L$ and $\bar{E} = E/L$

are vector bundles, so that we have an exact sequence $0 \longrightarrow \bar{E}_1 \longrightarrow \bar{E} \longrightarrow E_2 \longrightarrow 0$ and by the inductive hypothesis

$$c(E) = c(L)c(\bar{E}) = c(L)c(\bar{E}_1)c(E_2) = c(E_1)c(E_2)$$

□

By this theorem, we have an additive function with values in the multiplicative group of invertible formal series with coefficients in $A(X)$ for some cohomology theory A.

$$\begin{array}{rccc} f: & K(X) & \longrightarrow & A(X)[[t]] \\ & E & \longmapsto & 1 + c_1^A(E)t + \cdots + c_r^A(E)t^r + \cdots \end{array}$$

In this way, we obtain n-th Chern class $c_n^A \colon K(X) \to A(X)$.

3.1. **Chern roots.** Given a vector bundle $E \to X$, by splitting principle, there is a base change $\pi \colon X' \to X$ such that $\pi^*E = L_{\alpha_1} + \cdots + L_{\alpha_r}$ in $K(X')$ and $\pi^* \colon A(X) \to A(X')$ is injective. Note that $c_1^A(L_{\alpha_n}) = \alpha_n$ and $c(L_{\alpha_n}) = x - \alpha_n$, then by the above theorem, we have

$$c(\pi^*E) = (t - \alpha_1) \dots (t - \alpha_r)$$

we may call $\alpha_1, \dots, \alpha_r$ **Chern roots** of E and $c_n(E)$ is the n-th elementary symmetric function of the roots $\alpha_1, \dots, \alpha_r$,

$$c_n(E) = \sum_{i_1 < \cdots < i_n} \alpha_{i_1} \dots \alpha_{i_n}$$

For example, if we take A to be K-theory. Recall that the first Chern class of L is $1 - L^*$. Hence the first Chern class of vector bundle of rank r is

$$c_1^K(E) = (1 - L_{\alpha_1}^*) + \cdots + (1 - L_{\alpha_r}^*) = r - E^* \tag{7}$$

and

$$c_r^K(E) = (1 - L_{\alpha_1}^*) \cdot \cdots \cdot (1 - L_{\alpha_r}^*) = \sum_i (-1)^i \bigwedge^i E^*$$

Corollary 3.1. *The cohomology ring of the projective space* $\mathbb{P}^d$ *is*

$$A(\mathbb{P}^d) = A(pt)[x]/[x^{d+1}] = A(pt)[y]/[y^{d+1}]$$

where x_d *corresponds to* $c_1(\xi_d)$ *and* y *corresponds to* $y_d = c_1(\xi_d^*) = [\mathbb{P}^{d-1}]$

Proof. The projective space $\mathbb{P}^d$ is the projective bundle of trivial bundle of rank $d+1$ on a singe point pt. Hence the first identity is indeed true.

Note that in $\mathbb{P}^1$, we have exact sequence $0 \longrightarrow \xi_1 \longrightarrow 1 \oplus 1 \longrightarrow \xi_1^*$, where $1 \oplus 1$ is a trivial bundle of rank 2 on $\mathbb{P}^1$. Clearly, $y_1 = -x_1$. Now consider an embedding of projective line $i \colon \mathbb{P}^1 \to \mathbb{P}^d$, we must have $y_d = -x_d + a_2 x_d^2 + \cdots + a_d x^d$, since the first Chern class is functorial and $i^*\xi_d = \xi_1$, then $i^*(x_d) = x_1$, $i^*(y_d) = y_1$. Hence $1, y_d, \dots, y_d^d$ is a basis of $A(\mathbb{P}^d)$. □

Corollary 3.2. *Chern classes are always nilpotent.*

Proof. We just need to check the case of line invertible bundles, because all the Chern classes come from Chern roots.

Suppose $L \to X$ is a line bundle, then we apply Jouanolou's trick (see Appendix 5) on it so that there is an affine bundle $p \colon P \to X$ such that P is an affine variety. Thus p^*L is a line bundle on an affine variety and then is globally generated (very ample). Hence $p^*L = f^*(\xi_d^*)$ for some $f \colon P \to \mathbb{P}^d$. It follows that $p^*c_1(L) = f^*(y_d)$.

Since y_d is nilpotent, $p^*c_1(L)$ is nilpotent. Since p^* is an isomorphism, $c_1(L)$ is nilpotent. □

4. Universal Property of the K-theory

Theorem 4.1. *If a cohomology theory A follows the group law $x+y-xy$ of the K-theory, there is a unique morphism of cohomology theories $\varphi\colon K\to A$.*

Proof. Let $E\to X$ be a vector bundle on X, according to (7), we have $E=\mathrm{rk}(E)-c_1^K(E^*)$; hence the unique possible morphism $\varphi\colon K\to A$ is

$$\varphi(E):=\mathrm{rk}(E)-c_1^A(E^*)$$

because the first Chern class is functorial. We need to show that such φ is indeed a morphism between cohomology theories.

Now, we check it is a functor. Given $X\in\mathbf{smQProj}_k$, we need to show that $\varphi: K(X)\to A(X)$ is a ring homomorphism. Since Chern classes are additive (recall Theorem 3.4), it is an abelian group homomorphism clearly. To show φ is compatible with multiplication, it suffices to check φ preserves products of line bundles due to splitting principle. Since A follows the group law $x+y-xy$,

$$\begin{aligned}\varphi(L_1\otimes L_2)=&1-c_1^A(L_1^*\otimes L_2^*)\\ =&1-(c_1^A(L_1^*)+c_1^A(L_2^*)-c_1^A(L_1^*)c_1^A(L_2^*))\\ =&(1-c_1^A(L_1^*)(1-c_1^A(L_2^*)\\ =&\varphi(L_1)\varphi(L_2)\end{aligned}$$

It remains to show φ preserves direct images.

Note that φ preserves Chern classes of line bundles

$$\varphi(c_1^K(L))=\varphi(1-L^*)=1-\varphi(L^*)=1-(1-c_1^A(L))=c_1^A(L)$$

to finish the proof, we need Panin's lemma. □

4.1. Panin's lemma.

Lemma 4.2 (Panin's Lemma)**.** *Let A and $\bar{A}$ be two cohomology theories on $\mathbf{smQProj}_k$. If a natural transformation $\varphi\colon A\to\bar{A}$ preserves the first Chern class of the dual of the tautological line bundles $\xi_d\to\mathbb{P}^d$ i.e. $\varphi(c_1^A(\xi_d))=c_1^{\bar{A}}(\xi_d)$, then it preserves direct images:*

$$\varphi(f_*(a))=\bar{f}_*(\varphi(a)) \tag{8}$$

for any projective morphism $f\colon Y\to X$, and any $a\in A(Y)$.

Proof. Let $f\colon Y\to X$ be a projective morphism, then f is factored through $i\colon Y\hookrightarrow\mathbb{P}^d\times X$ and $pr_X\colon\mathbb{P}^d\times X\to X$, where i is a closed embedding and pr is the natural projection. It suffices to prove the case where f is a closed embedding or a natural projection $\pi_X\colon\mathbb{P}^n\times X\to X$.

To prove the lemma, we will use the technique of deformation to the normal cone in Section 6.

Step 1: If (8) holds for the zero section $s_0: Y\to\widetilde{N}=\mathbb{P}(C_YX\oplus\mathbb{1})$, the projective closure of the normal cone (note that in smooth case, the normal cone is the same as the normal bundle $\mathscr{N}_{Y/X}$), then it also holds for closed immersion $i\colon Y\to X$.

Proof. Let D' be the blow-up of $X\times\mathbb{A}^1$ along $Y\times\{0\}$, so that we have a commutative diagram

$$\begin{array}{ccccc} Y & \xhookrightarrow{\;i\;} & D'_1 \cong X & \longrightarrow & 1 \\ \big\downarrow{\scriptstyle i_1} & & \big\downarrow & & \big\downarrow \\ Y\times\mathbb{A}^1 & \xhookrightarrow{\;\iota\;} & D' & \xrightarrow{\;\rho\;} & \mathbb{A}^1 \\ {\scriptstyle i_0}\big\uparrow & & \big\uparrow & & \big\uparrow \\ Y & \hookrightarrow & D'_0 = (\widetilde{N}\cup Bl_Y X) & \longrightarrow & \{0\} \end{array}$$

Let $U = D' - (Y\times\mathbb{A}^1)$. By Axiom (4), we have a commutative diagram

$$\begin{array}{ccc} & & \bar{A}(U) \\ & & \big\uparrow{\scriptstyle \bar{j}^*} \\ \bar{A}(\widetilde{N}) & \xleftarrow{\;\bar{i}_0^*\;} & \bar{A}(D') \\ {\scriptstyle \bar{s}_{0*}}\big\uparrow & & \big\uparrow{\scriptstyle \bar{\iota}_*} \\ \bar{A}(Y) & \underset{\sim}{\xleftarrow{\;\bar{i}_0^*\;}} & \bar{A}(Y\times\mathbb{A}^1)) \end{array}$$

because $\widetilde{N}$ and $Y\times\mathbb{A}^1$ intersect transversally on $Y\times\{0\} = Y$ (their normal bundles are trivial bundles of rank 1). Note that $\bar{s}_0^*$ is injective because it is a section. Since right column of the diagram is an exact sequence, if $x\in\ker(\bar{j}^*)\cap\ker(\bar{i}_0^*)\subset\bar{A}(D')$, then there exists $y\in\bar{A}(Y\times\mathbb{A}^1)$ such that $x = \bar{\iota}^*(y)$. Further, $0 = \bar{i}_0^*(x) = \bar{i}_0^*\circ\bar{\iota}^*(y) = \bar{s}_0^*\circ\bar{i}_0^*(y)$, then $y = 0$, so is $x = 0$. Thus, we have $\ker(\bar{j}^*)\cap\ker(\bar{i}_0^*) = 0$.

Next, we consider another commutative diagram

$$\begin{array}{ccccc} & & \bar{A}(U) & & \\ & & \big\uparrow{\scriptstyle \bar{j}^*} & & \\ \bar{A}(\widetilde{N}) & \xleftarrow{\;\bar{i}_0^*\;} & \bar{A}(D') & \xrightarrow{\;\bar{i}_1^*\;} & \bar{A}(X) \\ {\scriptstyle \Psi_1}\big\uparrow & & \big\uparrow{\scriptstyle \Psi_2} & & {\scriptstyle \Psi_3}\big\uparrow \\ A(Y) & \underset{\sim}{\xleftarrow{\;i_0^*\;}} & A(Y\times\mathbb{A}^1)) & \underset{\sim}{\xrightarrow{\;i_1^*\;}} & A(Y) \end{array}$$

where $\mathbf{\Psi}_1 = \bar{s}_{0*}\circ\varphi-\varphi\circ s_{0*}$, $\mathbf{\Psi}_2 = \bar{\iota}_*\circ\varphi-\varphi\circ\iota_*$ and $\mathbf{\Psi}_3 = \bar{i}_*\circ\varphi-\varphi\circ i_*$. The morphism $\mathbf{\Psi}_1$ is zero by the assumption and so is $\mathbf{\Psi}_2$ because

$$\begin{aligned} \bar{i}_0^*\circ\mathbf{\Psi}_2 =& \bar{i}_0^*\circ(\bar{\iota}_*\circ\varphi-\varphi\circ\iota_*) \\ =&(\bar{i}_0^*\circ\bar{\iota}_*)\circ\varphi-\bar{i}_0^*\circ\varphi\circ\iota_* \\ =&(\bar{i}_0^*\circ\bar{\iota}_*)\circ\varphi-\varphi(i_0^*\circ\circ\iota_*) \\ =&(\bar{s}_{0*}\circ\bar{i}_0^*)\circ\varphi-\varphi(s_{0*}\circ i_0^*) \\ =&(\bar{s}_{0*}\circ\varphi\circ i_0^*)-\varphi(s_{0*}\circ i_0^*) \\ =&\mathbf{\Psi}_1\circ i_0^* = 0, \end{aligned}$$

$\bar{j}^*\circ\mathbf{\Psi}_2 = 0$ and $\ker(\bar{j}^*)\cap\ker(\bar{i}_0^*) = 0$. Then, $\mathbf{\Psi}_3 = \bar{i}_*\circ\varphi-\varphi\circ i_* = 0$ due to the commutative diagram. □

Step 2: The fundamental class of a hypersurface of an $X\in\mathbf{smQProj}_k$ is preserved by φ.

Proof. Let Y be a hypersurface a such X. Then by Jouanolou's trick, there is an affine bundle $\pi: A \to X$ such that A is affine and $\pi^{-1}(Y)$ is a hypersurface of A. There is an embedding $f: A \to \mathbb{P}^n$ and a hypersurface $\mathbb{P}^{n-1}$ corresponds to ξ_n such that $f^{-1}(\mathbb{P}^{n-1}) = \pi^{-1}(Y)$. Suppose L is the line bundle corresponding to the hypersurface Y in A, then π^*L is the line bundle corresponding to the hypersurface $\pi^{-1}(Y)$. Hence $\pi^*L = f^*\xi_n$. Since the first Chern class is functorial and φ preserves the first Chern class of ξ_n, φ also preserves the first Chern class of L. According Remark 2.2, the first Chern class of L is exactly $[Y]^A$, hence $\varphi([Y]^A) = [Y]^{\bar{A}}$. □

Step 3: Equation (8) holds for the zero section $s_0: Y \to \widetilde{E} = \mathbb{P}(E \oplus \mathbb{1})$ of the projective closure of any vector bundle $E \to Y$.

Proof. When $E = L$ is a line bundle, note that the zero section exhibits Y as a hypersurface of $\widetilde{L}$, then $\varphi(s_{0*}(1)) = \bar{s}_{0*}(1)$. Since s_0 is a section, $s_0^*: A(\widetilde{L} \to A(Y)$ is surjective. For any $a \in A(Y)$, there exists $b \in A(\widetilde{L})$ such that $a = s_0^*(b)$, then by projective formula

$$\varphi(s_{0*}(a)) = \varphi(s_{0*}(s_0^*(b))) = \varphi(bs_{0*}(1)) = \varphi(b)\bar{s}_{0*}(1) = \bar{s}_{0*}(\bar{s}_0^*(\varphi(b))) = \bar{s}_{0*}(\varphi(a))$$

For general case, we apply splitting principle again, so that we may assume the vector bundle E admits a filtration $\{E_i\}$ such that E_i/E_{i-1} are line bundles. Equation 8 holds for the zero section $Y \hookrightarrow \widetilde{E}_1$ and $\widetilde{E}_i \hookrightarrow \widetilde{E}_{i+1}$, hence it holds for the composition $s: Y \to \widetilde{E}$. □

Step 4: Equation (8) holds for the canonical projection $pr_X: \mathbb{P}^n \times X \to X$.

Proof. By Axiom (5), we just need to check the case where $p: \mathbb{P}^n \to \{pt\}$. Now we set

$$A = A(pt), \;\; x_n = c_1^A(\xi_n^*) = i_*(1) \in A(\mathbb{P}^n),$$
$$\bar{A} = \bar{A}(pt), \;\; \bar{x}_n = c_1^{\bar{A}}(\xi_n^*) = \bar{i}_*(1) \in \bar{A}(\mathbb{P}^n).$$

According to the hypothesis, we have $\varphi(x_n) = \bar{x}_n$, hence the homomorphism $\varphi: A(\mathbb{P}^n) \to \bar{A}(\mathbb{P}^n)$ induces an isomorphism of $\bar{A}$-algebras $\mathbb{A}(\mathbb{P}^n) \oplus_A \bar{A} \cong \bar{A}(\mathbb{P}^n)$.

We just need to check that the $\bar{A}$-linear map $\bar{p}_*: \bar{A}(\mathbb{P}^n) \to \bar{A}$ is obtained by base change of the A-linear map $p_*: A(\mathbb{P}^n) \to A$. Namely, the following diagram commutes

$$\begin{array}{ccc} A(\mathbb{P}^n) & \xrightarrow{\varphi} & \bar{A}(\mathbb{P}^n) \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle \bar{p}_*} \\ A & \xrightarrow{\varphi} & \bar{A} \end{array}$$

Let $\Delta_n = \Delta_*(1) \in A(\mathbb{P}^n \times \mathbb{P}^n) = A(\mathbb{P}^n) \otimes_A A(\mathbb{P}^n)$ be the fundamental class of the diagonal closed immersion $\Delta: \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$. Then we have

$$(p_* \times 1)(\Delta_n) = (p_* \times 1) \circ \Delta_*(1) = \mathrm{id}_*(1) = 1 \in A(\mathbb{P}^n)$$

where $p \times 1: \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ is the projection of the second component. Note that $p_* \in A(\mathbb{P}^n)^*$, the dual of $A(\mathbb{P}^n)$ and p_* is mapped to the unity by means of the following map

$$\begin{array}{cccc} f: & A(\mathbb{P}^n)^* & \longrightarrow & A(\mathbb{P}^n) \\ & \omega & \longmapsto & (\omega \otimes 1)(\Delta_n) \end{array}$$

Since we have shown that closed immersions are preserved by φ, such f is stable under φ. Namely $\bar{p}_*$ is mapped to the identity by $\bar{f}$. To show that p is preserved under φ, we just need to show that p_* is fully determined by f, which means the value of $(\omega\otimes 1)(\Delta_n)$ fully determines the ring homomorphism $(\omega\otimes 1)$. We claim that Δ_n is invertible in $A(\mathbb{P}^n)\otimes_A A(\mathbb{P}^n)$. To prove the claim, we may write the elements in $A(\mathbb{P}^n)\otimes_A A(\mathbb{P}^n)$ in the form of $(n+1)\times(n+1)$ matrices by

$$\sum_{r,s=0}^{n} a_{rs}x_n^r\otimes x_n^s \sim (a_{rs}) = \begin{bmatrix} a_{01} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nn} \end{bmatrix}$$

We may write an element in $\omega\in A(\mathbb{P}^n)^*$ by an $1\times(n+1)$ matrix

$$\omega \sim \begin{bmatrix} \omega_0 & \cdots & \omega_n \end{bmatrix}$$

which means $x_n^r\mapsto\omega_r$, and

$$(\omega\otimes 1)(\Delta_n) = \begin{bmatrix} \omega_0 & \cdots & \omega_n \end{bmatrix} \begin{bmatrix} a_{01} & \cdots & a_{0n} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_n^0 \\ \vdots \\ x_n^n \end{bmatrix}$$

To show p_* is fully determined by f, we just need to prove the corresponding matrix of Δ_n is non-singular. We will prove this assertion in the next step. □

Step 5: The matrix $\Delta_n\in A(\mathbb{P}^n)\otimes_A A(\mathbb{P}^n)$ of the diagonal is non-singular.

Proof. By induction on n, we prove that

$$\Delta_n = \sum_{r,s=0}^{n} a_{rs}x_n^r\otimes x_n^s = (a_{rs}) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & \bullet \\ 0 & \iddots & \iddots & \vdots \\ 1 & \bullet & \cdots & \bullet \end{bmatrix}$$

where $a_{rs}=0$ when $r+s<n$, and $a_{rs}=1$ when $r+s=n$. By projective formula,

$$i_*(x_{n-1}^r) = i_*i^*(x_n^r) = x_n^r\cdot i_*(1) = x_n^{r+1}$$

where $i\colon\mathbb{P}^{n-1}\to\mathbb{P}^n$, and

$$i^*(x_n) = i^*(c_1(\xi_n^*) = c_1(i^*\xi_n^*) = c_1(\xi_{n-1}^*) = x_{n-1}$$

Then we consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}^n & \xleftarrow{\quad i \quad} & \mathbb{P}^{n-1} \\ & & \downarrow{\scriptstyle\Delta} \\ \downarrow{\scriptstyle\Delta} & & \mathbb{P}^{n-1}\times\mathbb{P}^{n-1} \\ & & \downarrow{\scriptstyle 1\times i} \\ \mathbb{P}^n\times\mathbb{P}^n & \xleftarrow{\quad i\times 1 \quad} & \mathbb{P}^{n-1}\times\mathbb{P}^n \end{array}$$

then by Axiom (4), we have

$$\begin{array}{ccc} A(\mathbb{P}^n) & \xrightarrow{\quad i^* \quad} & A(\mathbb{P}^{n-1}) \\ & & \downarrow{\scriptstyle\Delta_*} \\ \downarrow{\scriptstyle\Delta_*} & & A(\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}) \\ & & \downarrow{\scriptstyle 1\times i_*} \\ A(\mathbb{P}^n\times\mathbb{P}^n) & \xrightarrow{\quad i^*\times 1 \quad} & A(\mathbb{P}^{n-1}\times\mathbb{P}^n) \end{array}$$

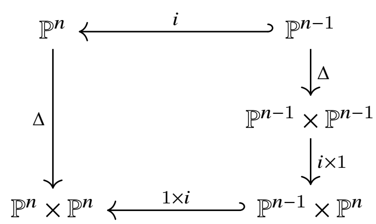
In particular, $(1\times i_*)\circ\Delta_*\circ i^*(1) = (i^*\times 1)\circ\Delta_*(1)$. Let $\Delta_{n-1} = \sum_{r,s=0}^{n-1} a'_{rs}(x_{n-1}^r \otimes x_{n-1}^s)$. Note that

$$(1\times i_*)\circ\Delta_*\circ i^*(1) = (1\times i_*)(\Delta_{n-1}) = (1\times i_*)(\sum_{r,s=0}^{n-1} a'_{rs}x_{n-1}^r \otimes x_{n-1}^s) = \sum_{r,s=0}^{n-1} a'_{rs}x_{n-1}^r \otimes x_n^{s+1}$$

$$(i^*\times 1)\circ\Delta_*(1) = (i^*\times 1)(\Delta_n) = (i^*\times 1)(\sum_{r,s=0}^{n} a_{rs}x_n^r \otimes x_n^s) = \sum_{r,s=0}^{n} a_{rs}x_{n-1}^r \otimes x_n^s$$

then we have $\sum_{r,s=0}^{n-1} a'_{rs}x_{n-1}^r \otimes x_n^{s+1} = \sum_{r,s=0}^{n} a_{rs}x_{n-1}^r \otimes x_n^s$, which implies $a'_{r,s-1} = a_{rs}$. Hence, by the inductive hypothesis, for $s > 0$, $a_{rs} = 1$, when $r+s=n$ and $a_{rs} = 0$, when $r+s<n$.

Similarly, when we consider the diagram,



and by the symmetry of r and s, we have for $r > 0$, $a_{rs} = 1$, when $r+s=n$ and $a_{rs} = 0$, when $r+s<n$. □

Combine these steps, we finish the proof of Panin's lemma. □

4.2. **Compute possible direct images of a cohomology theory.** Notation: Given a formal series $F(t) \in A(pt)[[t]]$, if $\alpha_1,\dots,\alpha_n$ are Chern roots of a vector bundle E, then the **additive extension** is

$$F_+(E) = F(\alpha_1) + \cdots + F(\alpha_n)$$

the **multiplicative extension** is

$$F_\times(E) = F(\alpha_1)\dots F(\alpha_n)$$

Since both F_+ and $F_\times$ are invariant under any permutation of the Chern roots, hence they power series in the elementary symmetric functions of Chern roots, which are just the Chern classes of E.

Theorem 4.3. *Let A be a contravariant functor from* **smQProj**$_k$ *to* **Cring**, *f_* be a class of direct images such that (A, f_*) is a cohomology theory(its Chern classes are denoted by c_1), and f_*^{new} be another class of direct images for A such that (A, f_*^{new}) is another cohomology theory(its Chern classes are denoted by c_1^{new}). For any projective morphism $f: Y \to X$, denote $T_f := T_Y - f^*T_X \in K(Y)$ the virtual relative tangent bundle. Then there exists an invertible formal series $F(t) \in A(py)[[t]]$ such that*

$$f_*^{new}(a) = f_*(F_\times(T_f)^{-1}\cdot a).$$

Moreover, every invertible series defines new direct images in this way.

Proof. According to Corollary 3.1,

$$A(\mathbb{P}^d) = A(pt)[c_1(\xi_d)]/(c_1(\xi_d)^{d+1}) = A(pt)[c_1^{new}(\xi_d)]/(c_1^{new}(\xi_d)^{d+1}).$$

Suppose

$$c_1^{new}(\xi_d) = b_0 + b_1\cdot c_1(\xi_d) + \cdots + b_d\cdot c_1(\xi_d)^d,$$

and similarly,

$$c_1^{new}(\xi_{d+1}) = b'_0 + b'_1 \cdot c_1(\xi_{d+1}) + \cdots + b'_{d+1} \cdot c_1(\xi_{d+1})^{d+1},$$

and let $i\colon \mathbb{P}^d \hookrightarrow \mathbb{P}^{d+1}$ be the closed embedding of hyperplane, then

$$i^* c_1^{new}(\xi_{d+1}) = c_1^{new}(i^*\xi_{d+1}) = c_1^{new}(\xi_d) = b_0 + b_1 \cdot c_1(\xi_d) + \cdots + b_d \cdot c_1(\xi_d) =$$

hence

$$b_0 + b_1 \cdot c_1(\xi_d) + \cdots + b_d \cdot c_1(\xi_d)^d = b'_0 + b'_1 \cdot c_1(\xi_{d+1}) + \cdots + b'_{d+1} \cdot c_1(\xi_{d+1})^d$$

Hence $b_i = b'_i$ for $(i < d+1)$. In this way, we may construct a series $f(t) = b_0 + b_1 t + b_2 t^2 + \dots$, such that

$$c_1^{new}(\xi_n) = f(c_1(\xi_n)),$$

for all $n \in \mathbb{N}$.

For $d = 0$, $\mathbb{P}^d = pt$, hence $b_0 = 0$. For $d = 1$, consider the closed embedding $i\colon pt \to \mathbb{P}^1$ and the open complement $\mathbb{P}^1 \setminus pt = \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Then by the Axiom (3), we have

$$A(pt) \xrightarrow{i_*^{new}} A(\mathbb{P}^1) \xrightarrow{j^*} A(\mathbb{A}^1),$$

$$A(pt) \xrightarrow{i_*} A(\mathbb{P}^1) \xrightarrow{j^*} A(\mathbb{A}^1).$$

Then $\ker j^* = A(pt)c_1(\xi_1) = A(pt)c_1^{new}(\xi_1)$. Hence $c_1(\xi_1) = b_1 c_1^{new}(\xi_1)$ for some invertible element $b_1 \in A(pt)$. Now we set $F(t) = b_1 + b_2 t + \dots$, and note that

$$c_1^{new}(\xi_d) = c_1(\xi_d)F(c_1(\xi_d)). \tag{9}$$

For any projective morphism $f : Y \to X$, consider the map $f_*(F_\times(T_f)^{-1}\bullet) : A(Y) \to A(X)$. The pair $(A, f_*(F_\times(T_f)^{-1}\bullet))$ is indeed a cohomology theory. The Axioms except Axiom (6) are easy to check. Now we check it for Axiom (6). Let $x = x_E \in A(\mathbb{P}(E))$, and $y = x_E^{new} = xF(x) = b_1 x + \cdots +$. Note that the powers of x forms a basis of free module $A(\mathbb{P}(E))$ and the powers of y also form a basis of free module $A(\mathbb{P}(E))$ clearly.

Now we compute the fundamental classes in $(A, f_*(F_\times(T_f)^{-1}\bullet))$: for a closed embedding $i\colon Y \to X$:

$$i_*(F_\times(\mathscr{N}_{Y/X}) \cdot 1) = i_*((F_\times(i^*\mathscr{N}_{Y/X})) = F_\times(L_Y) i_*(1) = [Y] \cdot F([Y])$$

where L_Y is the sheaf defined by the hypersurface in Remark 2.2, Equation (5).

Therefore the first Chern class of line bundle L is $s_0^* s_{0*}(F_\times(s_0^* L) \cdot 1) = c_1(L)F(c_1(L))$.

Consider the identity natural transformation $A \to A$ and by Panin's lemma, it gives an isomorphism between cohomology theories $(A, f_*^{new}) \to (A, f_*(F_\times(T_f)^{-1}\bullet))$, and we have $f_*^{new} = f_*(F_\times(T_f)^{-1}\bullet)$.

Conversely, we can define a new class of direct images in this way for any invertible formal series. □

4.3. **Grothendieck-Riemann-Roch.** Let A be a cohomology theory following the additive law, we need to modify it such that it is turned to be a cohomology theory following the multiplicative group law.

The ideal is to consider $A \otimes \mathbb{Q}$, because we may modify the direct image of $A \otimes \mathbb{Q}$ with an exponential so that the new theory follows the multiplicative law. Note that exponential is a formal series with coefficients in $\mathbb{Q}$.

Since $e^{at} = (1 - (1 - e^{at}))$, we must fix a formal series $F(t)$ such that

$$c_1^{new}(L_x) = 1 - e^{ax}$$

(Recall Theorem 4.3). Then according to equation 9, we have

$$1 - e^{ax} = xF(x)$$

Now $1 - e^{at} = -at + \cdots$, so we may fix $a = -1$. To transform the additive law of $A \otimes \mathbb{Q}$ into a multiplicative law, we modify it with the formal series

$$F(t) = \frac{1 - e^{-t}}{t} = 1 - \frac{t}{2!} + \frac{t^2}{3!} + \cdots$$

via Theorem 4.3, so that the new cohomology theory $A^{new} = (A \otimes \mathbb{Q}, f_*^{new})$ follows the multiplicative law $x+y-xy$. By the universal property of K-theory, there exists a unique morphism of cohomology theories

$$ch\colon K \to A \otimes \mathbb{Q}$$

by

$$ch(L_x) = 1 - c_1^{new}(L_x^*) = 1 - c_1^{new}(L_{-x}) = 1 - (1 - e^x) = e^x$$

If we consider the multiplicative extension Td of the series

$$F(t)^{-1} = \frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{720} + \cdots,$$

which is usually called **Todd class**, then we obtain

Theorem 4.4 (Grothendieck-Riemann-Roch). *Let A be a cohomology theory following the additive law. For any projective morphism $f : Y \to X$, we have the following commutative square.*

$$\begin{array}{ccc} K(Y) & \xrightarrow{f_!} & K(X) \\ \downarrow{\scriptstyle Td(T_Y)\cdot ch} & & \downarrow{\scriptstyle Td(T_X)\cdot ch} \\ A(Y)\otimes\mathbb{Q} & \xrightarrow{f_*} & A(X)\otimes\mathbb{Q} \end{array}$$

Proof. Since $ch\colon K \to A_{\mathbb{Q}}^{new}$ preserves direct images,

$$ch(f_!(y)) = f_*^{new}(ch(y)) = f_*[F(f^*T_X - T_Y)ch(y)] \tag{10}$$

$$= F(T_X) f_*[F(T_Y)^{-1} ch(y)] \tag{11}$$

$$= Td(T_X)^{-1} f_*[Td(T_Y) ch(y)] \tag{12}$$

□

For a vector bundle E with Chern roots $\alpha_1, \ldots, \alpha_n$, we have

$$ch(E) = \sum_i e^{\alpha_i} = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \cdots$$

$$Td(E) = \prod_i (1 + \frac{\alpha_i}{2} + \frac{\alpha_i^2}{12} + \cdots) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots$$

Note that Ω_X is the dual bundle of T_X and $c_1(\Omega_X)$ is the **canonical divisor** on X, denoted by K usually. We let $X = pt = \operatorname{Spec} k$ and $Y = C$, where C is a smooth irreducible projective curve over k, and let $\pi\colon C \to pt$ be the structure map. Then we have

$$\chi(C, E) = \pi_!(E) = \pi_*(Td(T_C) \cdot ch(E)) = \deg c_1(E) - \frac{r}{2} \deg K$$

If we take $E = \mathscr{O}_X$, then we have

$$\chi(C, \mathscr{O}_X) = 1 - g = -\frac{1}{2}\deg K$$

namely $\deg K = 2g - 2$.

If we take $E = \mathscr{O}_X(D)$ for divisor D, then we have

$$\chi(C, \mathscr{O}_X(D)) = \deg D - \frac{1}{2}\deg K = \deg D + 1 - g.$$

5. Appendix: Jouanolou's Trick

The main reference of this appendix is [Aso09].

Theorem 5.1 (Jouanolou)**.** *Given a quasi-projective variety X over a k, there exists a pair $(\tilde{X}, \pi)$, where $\tilde{X}$ is an affine scheme, smooth over k, and $\pi\colon \tilde{X} \to X$ is a Zariski locally trivial smooth morphism with fibers isomorphic to affine spaces.*

Let Q_{2m-1} denote the closed subscheme of $\mathbb{A}^{2m}$ (with coordinate $x_1, \ldots, x_{2m}$) cut by the equations

$$\sum_i x_i x_{d+i} = 1$$

Lemma 5.2. *For any $m \geq 1$, the projection onto $x_1, \ldots, x_m$ determines a morphism*

$$\varphi\colon Q_{2m-1} \to \mathbb{A}^m \setminus 0$$

is Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{m-1}$. In particular, when $m = 1$, it is an isomorphism.

Proof. We have an affine open cover $\{D(x_i)\}_i^m$ of $\mathbb{A}^m \setminus 0$. For each $D(x_j) \cong \operatorname{Spec} k[x_1, \ldots, x_n]_{x_j}$,

$$\varphi^{-1}(D(x_i)) \cong \operatorname{Spec}\,(k[x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m}]/\sum_i x_i x_{m+i} - 1)_{x_j}$$

We can write the R.H.S into

$$\operatorname{Spec} k[x_1, \ldots, x_m, x_{m+1}, \ldots, x_{2m}]_{x_j}/(x_{m+j} - x_j^{-1}(1 + \sum_{i \neq j} x_i x_{m+i}))$$

More concisely, it is

$$\operatorname{Spec} k[x_1, \ldots, x_m, \ldots, \hat{x_{m+j}}, \ldots, x_{2m}]_{x_j}$$

where

$$k[x_1, \ldots, x_m, \ldots, \hat{x_{m+j}}, \ldots, x_{2m}]_{x_j} \cong k[x_1, \ldots, x_m]_{x_j} \otimes_k k[x_{m+1}, \ldots, \hat{x_{m+j}}, \ldots, x_{2m}]$$

which is what we need. ◻

The construction of Q_{2m-1} help us to construct an affine bundle that is affine on any open subset of $\mathbb{A}^n$.

Proposition 5.1. *Suppose $Z \subset \mathbb{A}^n$ is a closed subscheme cut by $f_1, \ldots, f_d$. Consider the morphism $\mathbb{A}^n \to \mathbb{A}^d$ defined by the functions $f_1, \ldots, f_d$. Defined a morphism*

$$F\colon U = \mathbb{A}^n \setminus Z \to \mathbb{A}^d \setminus 0$$

via $f_1, \dots, f_d$. Then the fiber product $\widetilde{U} := D \times_{\mathbb{A}^d \setminus 0} Q_{2d-1}$ via F and φ is isomorphic to the closed subscheme of $\mathbb{A}^n \times \mathbb{A}^d$ (with coordinate $y_1, \dots, y_n, x_1, \dots, x_d$) cut by the equation

$$\sum_i x_i f_i(y_1, \dots, y_n)$$

and is in particular affine. Moreover, the projection $\pi \colon \widetilde{U} \to U$ induced by the pull-back is a Zariski locally trivial with fibers isomorphic to $\mathbb{A}^{d-1}$, and is in particular smooth.

Proof. Note that F is an affine morphism, since $F^{-1}(D(x_j) = D(f_j)$ for each j. According to the definition of fiber products, we have

$$\pi^{-1}(D(f_j)) \cong D(f_j) \times_{D(x_j)} \varphi^{-1}(D(x_i)) \tag{13}$$

$$= \operatorname{Spec} k[y_1, \dots, y_n]_{f_j} \otimes_{k[x_1,\dots,x_d]_{x_j}} k[x_1, \dots, x_{2d}]_{x_j} / \sum_i x_i x_{d+i} - 1 \tag{14}$$

where $k[y_1, \dots, y_n]_{f_j}$ is a $k[x_1, \dots, x_d]_{x_j}$ via $F : x_i \mapsto f_i$. By some simple algebraic cancellation (replace x_i by f_i), it is

$$\operatorname{Spec} k[y_1, \dots, y_n, f_1, \dots, f_d, x_{d+1}, \dots, x_{2d}]_{f_j} / \sum_i f_i x_{d+i} - 1$$

More concisely

$$\operatorname{Spec} k[y_1, \dots, y_n, x_{d+1}, \dots, x_{2d}]_{f_j} / \sum_i f_i x_{d+i} - 1$$

and there is no harm to replace x_{d+i} by x_i as coordinate.

In this way, $\widetilde{U} \to Q_{2d-1}$ is an affine morphism (note that affine morphisms are preserved by base change) and in particular, $\widetilde{U}$ is affine. Moreover, it is indeed the closed in $\mathbb{A}^d \times \mathbb{A}^n$ cut by $\sum_i x_i f_i - 1$ and it is a Zariski locally trivial fibration with fiber isomorphic to $\mathbb{A}^n \times \mathbb{A}^{d-1}$. □

Now we have shown that for any open set of $\mathbb{A}^n$, we can find such affine bundle in Jouanolou's lemma. For any locally closed subscheme of $\mathbb{A}^n$ or quasi-affine variety, it is still true.

Let V be a finite dimensional k-vector space with basis $v_1, \dots, v_n$ and $V^\vee$ be its dual space. Let $v : V \times V^\vee \to k$ denote the map of the valuation map. Let $\mathbb{P}(V) = \operatorname{Proj} \operatorname{Sym}^\bullet V^\vee$ and note that $\operatorname{Sym}^\bullet V^\vee = k[x_1, \dots, x_n]$, where x_i are the dual basis of $v_1, \dots, v_n$. Then we consider the hypersurface H of $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ cut by the map v. (Note that a point in $\mathbb{P}(V)$ is a subspace of V of dimension 1, hence

$$H = \{[v] \times [x] \in \mathbb{P}(V) \times \mathbb{P}(V^\vee) \mid x(v) = 0\}$$

is well-defined.

Proposition 5.2. *Let $\widetilde{\mathbb{P}(V)}$ be $\mathbb{P}(V) \times \mathbb{P}(V^\vee) \setminus H$, then it is affine and the composition morphism*

$$\widetilde{\mathbb{P}(V)} \hookrightarrow \mathbb{P}(V) \times \mathbb{P}(V^\vee) \to \mathbb{P}(V)$$

is a Zariski locally trivial affine morphism with fibers isomorphic to affine space.

Note that $\mathbb{P}(V^\vee)$ is the dual projective space of $\mathbb{P}(V)$.

Proof. Explicitly,

$$H = \{[x_1 : \cdots : x_n] \times [y_1 : \cdots : y_n] \mid \sum x_i y_i = 0\}$$

By Serge embedding, $\mathbb{P}(V) \times \mathbb{P}(V^\vee)$ is a closed subvariety $\mathbb{P}^{n^2-1}$. Recall the Serge morphism is

$$([x_1 : \cdots : x_n], [y_1 : \cdots : y_n]) \mapsto [x_0 y_0 : \cdots : x_i y_j : \cdots : x_n y_n]$$

and let $[\cdots : Z_{ij} : \dots]$ be the projective coordinate,

$$H = V(\sum Z_{ii}) \cap \mathbb{P}(V) \times \mathbb{P}(V^\vee)$$

and note that the complement of H in $\mathbb{P}^{n^2-1}$ is affine. Then

$$\widetilde{\mathbb{P}(V)} = \mathbb{P}(V) \times \mathbb{P}(V^\vee) \cap (\mathbb{P}^{n^2-1} \setminus H)$$

which is a closed subvariety of an affine variety and is in particular affine.

The morphism from $\widetilde{\mathbb{P}(V)}$ to $\mathbb{P}(V)$ is a composition of open embedding and projection, so it is smooth clearly.

Next, we need to show each fiber is an affine space. For $[x] \in \mathbb{P}(V)$, let $H_{[x]}$ be the restriction of H at $[x]$. Then $H_{[x]}$ is clearly a hyperplane of $\mathbb{P}(V^\vee)$ according to our construction of H. Then the fiber at $[x]$ is $\mathbb{P}(V^\vee) \setminus H_{[x]}$, which is an affine space clearly. □

According to this proposition, every projective variety satisfies the Jouanolou's theorem clearly. Then combine Proposition 5.1 and Proposition 5.2, we can prove the case for all quasi-projective varieties over k.

Proof of Jouanolou's lemma. Note that a quasi-projective variety X is a locally closed subset of some projective space $\mathbb{P}^n$. Let $\pi\colon A \to X$ be an affine bundle of $\mathbb{P}^n$ and by Proposition 5.2, we may assume A is affine. Then let $P = \pi^{-1}(X)$ $\pi\colon P \to X$ is an affine bundle on X and P is a locally closed subvariety in A. By Proposition 5.1, there is an affine bundle $\pi'\colon Q \to P$ where Q is affine. Clearly, $\pi' \circ \pi : Q \to X$ exhibits Q as an affine bundle on X. □

6. Apppendix: Deformation to the Normal Cone

6.1. **Normal cones.** Suppose $i : Y \to X$ is a closed embedding of smooth quasi-projective varieties over k. Let $\mathscr{N}_{Y/X}$ be the normal bundle of Y with respect to the embedding i. Let $s_0\colon Y \to \mathscr{N}_{Y/X}$ be the zero section of the normal bundle and in this way we identify Y as a subvariety of $\mathscr{N}_{Y/X}$.

Suppose $\mathscr{I}_Y$ is the ideal of Y on X, then the normal cone is $\mathbf{Spec}_X(\bigoplus_i \mathscr{I}_Y^i/\mathscr{I}_Y^{i+1}))$ and denoted by $C_Y X$. Note that $\mathscr{I}_Y^0 = \mathscr{O}_X$ and we may identify $\mathbf{Spec}_X(\mathscr{O}_X/\mathscr{I}_Y)$ with Y. Hence the inclusion $\mathscr{O}_X/\mathscr{I}_Y \hookrightarrow \bigoplus_i \mathscr{I}_Y^i/\mathscr{I}_Y^{i+1}$ determines a projection:

$$p\colon C_Y X \to Y$$

and the quotient map $\bigoplus_i \mathscr{I}_Y^i/\mathscr{I}_Y^{i+1} \to \mathscr{O}_X/\mathscr{I}_Y$ determines the zero section (since they are smooth, the normal cone is the normal bundle, vice versa)

$$s_0\colon Y \to C_Y X$$

Note that if Y is a single point, then the normal cone $C_Y X$ is the tangent cone of Y in X.

Further, the **projective normal cone** $\mathbb{P}(C_Y X)$ is $\mathbf{Proj}_X(\bigoplus_i \mathscr{I}_Y^i/\mathscr{I}_Y^{i+1})$.

Next we show the relations between these notions and blow-ups. Let $Bl_Y X$ be the blow up on X along Y. The construction of the blow up is given by

$$Bl_Y X := \mathbf{Proj}_X(\bigoplus_{i=0}^{\infty} \mathscr{I}_Y^n)$$

and there is a natural projection

$$\pi : Bl_Y X \to X$$

determined by $\mathscr{O}_X \hookrightarrow \bigoplus_{i=0}^{\infty} \mathscr{I}_Y$. We may also view X as a subvariety of $Bl_Y X$. Now we claim that $\pi^{-1}(Y) \cong \mathbb{P}(C_Y X)$, since

$$\bigoplus_{i=0}^{\infty} \mathscr{I}_Y \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}_Y \cong \bigoplus_i \mathscr{I}_Y^{i-1}/\mathscr{I}_Y^i$$

where we view $\pi^{-1}(Y)$ as the pull-back of $Y \hookrightarrow X$ along $Bl_Y X \to X$ and the definition of pull-back gives

$$\mathbf{Proj}_X(\bigoplus_{i=0}^{\infty} \mathscr{I}_Y^n) \times_{\mathbf{Spec}_X(\mathscr{O}_X)} \mathbf{Spec}_X(\mathscr{O}_X/\mathscr{I}_Y) = \mathbf{Proj}_X(\bigoplus_{i=0}^{\infty} \mathscr{I}_Y \otimes_{\mathscr{O}_X} \mathscr{O}_X/\mathscr{I}_Y)$$

Hence the blow-up $Bl_Y X$ is a union of X and $\mathbb{P}(C_Y X)$ along Y.

Example 6.1. Suppose X is affine and the ideal corresponds to Y is $I = (f_1, \dots, f_d)$. Then the blow-up is a subvariety of $X \times \mathbb{P}^{d-1}$ defined by the kernel of the canonical homomorphism from $A[X_1, \dots, X_d]$ to $\bigoplus_r I^r$ sending X_i to f_i. If $(f_1, \dots, f_d)$ is a regular sequence, then $Bl_Y X$ is cut by equations $X_i f_j - X_j f_i$ for $i < j$. In general, $Bl_Y X$ is the scheme-theoretic closure of the graph of the morphism from $X \setminus Y$ to $\mathbb{P}^{d-1}$ defined by $f_1, \dots, f_d$.

6.2. **Deformation.** Actually, the normal bundle or the normal cone shows how Y can "move" or "deformation" on X. More specifically, a section to $\mathscr{N}_{Y/X}$ is an infinitesimal deformation of Y. To state the phenomenon more precisely, we need to introduce the concept of **Hilbert scheme**. Roughly speaking, a Hilbert scheme H of X(we may assume $X = \mathbb{P}^n$) is the parameter space of closed subschemes, namely, every point in H represents a closed subscheme in X. In this view point, $[Y]$ is a point in $[Y]$ and the tangent space $T_{[Y]}H$ of $[Y]$ in H is $\Gamma(Y, \mathscr{N}_{Y/X})$. This is not the main goal in this survey, I just want to use Hilbert scheme to show some intuition of normal cones. Further, we may view a deformation from a closed subscheme Y to Y' as a "path" in H from point $[Y]$ and $[Y']$. The material to build the path is $\mathbb{A}^1$, just as the role of $\mathbb{R}^1$ in topology and differential geometry.

More concretely, since we may identify Y with a subvariety of $\mathscr{N}_Y X$ by the zero section, for any section $s : Y \to \mathscr{N}_Y X$, we have

$$\begin{array}{cccc} s_t : & Y \times \mathbb{A}_k^1 & \longrightarrow & C_Y X \\ & (a, \lambda) & \longmapsto & (a, \lambda s(a)) \end{array}$$

This morphism shows how Y deform in $C_Y X$ along the direction $s \in \Gamma(Y, \mathscr{N}_Y X)$. When λ is infinitesimal, $s_\lambda(Y)$ is almost a subvariety in X (However, we cannot say $s_\lambda(Y)$ is in X, so there is an "almost").

Although the normal bundle is good to describe the infinitesimal deformation, it is just the infinitesimal case and it is not precise enough to describe an actual deformation because there is an "almost"! Therefore, we need to find a way to delete the word "almost" and describe the deformation precisely.

To fix the gap, we need a *deformation* from the given embedding $Y \hookrightarrow X$ to the zero section $Y \hookrightarrow C_Y X$. More precisely, the word *deformation* means that there is a variety D with the following diagram

$$\begin{array}{ccc} Y \times \mathbb{A}^1 & \hookrightarrow & D \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \mathbb{A}^1 & \end{array}$$

such that over $t \neq 0$, the embedding from $Y \times \{t\}$ to $D_t = \rho^{-1}(t)$ is isomorphic the given embedding $Y \hookrightarrow X$, namely, we have the following commutative diagram

$$\begin{array}{ccc} Y \times \{t\} & \xrightarrow{\sim} & Y \\ \big\downarrow & & \big\downarrow i \\ D_t & \xrightarrow{\sim} & X \end{array}$$

and $Y \times \{0\} \hookrightarrow D_0$ is isomorphic to the zero section from Y to the normal cone $C_Y X$. We may denote such D by $D_Y X$ to indicate that the construction is depended on the embedding $Y \hookrightarrow X$. When the embedding is clear, we use D to represent $D_Y X$ simply. We may call it the **deformation space** of $Y \hookrightarrow X$.

Now we try to construct such deformation space. First to consider the affine case. Suppose X is affine with coordinate ring A and Y is given by an ideal $I = (f_1, \dots, f_d)$. Let D' be the scheme-theoretic closure of the graph of the morphism

$$X \setminus Y \times \mathbb{G}_m \longrightarrow \mathbb{P}^d$$

where $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ and the morphism is defined by $(P, t) \mapsto [f_1(P) : \cdots : f_d(P) : t]$. Thus D' is a closed subvariety of $X \times \mathbb{A}^1 \times \mathbb{P}^d$. Note that $Y \times \mathbb{A}^1$ is embedded in D' by

$$Y \times \mathbb{A}^1 = Y \times \mathbb{A}^1 \times \{[0 : \cdots : 0 : 1]\} \subset D'$$

Over $t = 0$, D'_0 contains the blow-up $Bl_Y X$ (recall Example 6.1). However, this is disjoint from $Y \times \{0\}$. We shall see that the complement to $Bl_Y X$ in D'_0 is the normal cone $C_Y X$, so that $D' \setminus Bl_Y X = D$ is what we need.

Refer to [Ger64], there is an algebraic version of deformation coincident with the definition of deformation spaces and this help us check some facts we have not proven before.

We define the graded ring B by

$$B = \cdots \oplus I^n T^{-n} \oplus \cdots \oplus I T^{-1} \oplus A \oplus AT \oplus \cdots \oplus AT^n \oplus \dots$$

i.e. $B = \bigoplus_{n=-\infty}^{\infty} I^n T^n$ where $I^m = A$ for $m \leq 0$, and T is an indeterminate. One may view D as the affine variety whose coordinate ring is B. The projection from D to $\mathbb{A}^1$ corresponds to the canonical inclusion from $k[T]$ to B, and the embedding of $Y \times \mathbb{A}^1$ is the canonical surjection $B \to A/I[T]$. Since the canonical homomorphism from $A[T]$ to B becomes an isomorphism after inverting T by localization i.e. $B_T \cong A[T]_T$. Hence the embedding $Y \times \mathbb{A}^1 \subset D \setminus D_0$ is isomorphic to the trivial embedding of $Y \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$. Over $T = 0$, since

$$B/TB \cong \bigoplus_{n=0}^{\infty} I^n / I^{n+1}$$

we see that $D_0 = C_Y X$, with Y embedded as the zero section.

In fact, the construction of D' is the blow-up of $X \times \mathbb{A}^1$ along $Y \times \{0\}$

$$Bl_{Y\times\{0\}} X \times \mathbb{A}^1 = \mathrm{Proj}(A[T] \oplus (I,T) \oplus (I,T)^2 \oplus (I,T)^3 \oplus \dots)$$

The normal cone to $Y \times \{0\} \hookrightarrow X \times \mathbb{A}^1$ is the cone

$$C_Y X \oplus \mathbb{1} = \mathrm{Spec}\,(\bigoplus_n I^n/I^{n+1} \otimes_{A/I} A/I[T])$$

and the exceptional divisor is

$$\mathbb{P}(C_Y X \oplus \mathbb{1}) = \mathrm{Proj}(\bigoplus_n I^n/I^{n+1} \otimes_{A/I} A/I[T])$$

which is the projective closure of $C_Y X$.(Note that the projective closure of $\mathbb{A}^n =$ Spec $(k[X_1, \dots, X_n]$ is $\mathbb{P}^n = \mathrm{Proj}(k[X_1,\dots,X_n] \otimes k[T]$ and the intuition is to paste a $\mathbb{P}^{n-1}$ with $\mathbb{A}^n$ to get $\mathbb{P}^n$. For general vector bundle $E \to X$, we proceed this process affine locally and the projective closure is indeed $\mathbb{P}(E \oplus \mathbb{1})$.) Hence, the blow-up $Bl_Y X$ is contained in D' clearly. Let $\rho\colon D' \to \mathbb{A}^1$ be the canonical projection given by $k[T] \to \bigoplus_n I^n/I^{n+1} \otimes_{A/I} A/I[T]$ and the fiber

$$D'_0 = \mathrm{Proj}(A[T] \oplus (I,T) \oplus (I,T)^2 \oplus (I,T)^3 \oplus \cdots)_T = \mathrm{Proj} B$$

at 0 is a union of $Bl_Y X = \mathrm{Proj} B/TB$ and $\mathbb{P}(C_Y X \oplus \mathbb{1}) = \mathrm{Proj} B/IB$, where the intersect is $\mathbb{P}(C_Y X) = \mathrm{Proj}(\bigoplus_n I^n/I^{n+1})$ clearly. Hence we have the following commutative diagram

$$\begin{array}{ccccc}
Y & \hookrightarrow & D'_1 \cong X & \longrightarrow & 1 \\
\downarrow{\scriptstyle i_1} & & \downarrow & & \downarrow \\
Y \times \mathbb{A}^1 & \hookrightarrow & D' & \xrightarrow{\rho} & \mathbb{A}^1 \\
{\scriptstyle i_0}\uparrow & & \uparrow & & \uparrow \\
Y & \hookrightarrow & D'_0 = \mathbb{P}(C_Y X \oplus \mathbb{1}) \cup Bl_Y X & \longrightarrow & \{0\}
\end{array}$$

In particular, $D' \setminus Bl_Y X$ coincides with the previous construction of D.

We have already solved the problem in affine case perfectly. In general, according to the the powerful lemmas 27.2.1 in Stacks Project, it is not hard to give a relative construction just like $\mathbf{Spec}_X$ or $\mathbf{Proj}_X$. Specifically, the general construction of $D'_X Y$ is exactly the blow-up on $X \times \mathbb{A}^1$ along $Y \times \{0\}$, where X and Y are not necessary to be affine.

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SUSTech
Email address: 11711505@mail.sustech.edu.cn